

# HIGHER ORDER DERIVED FUNCTORS

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**ABSTRACT.** We define a notion of higher derived functors, and show how these appear as the  $E^r$ -term of many commonly used spectral sequences, including the (stable or unstable) Adams spectral sequence.

## 0. INTRODUCTION

Many of the familiar spectral sequences of algebraic topology are constructed as the homotopy spectral sequence of a simplicial or cosimplicial space: for example, the spectral sequence of a double complex, the (stable or unstable) Adams spectral sequence, the Eilenberg-Moore spectral sequence, and even the Serre spectral sequence (see [D]). Moreover, the  $E^2$ -term of such spectral sequences can often be identified as derived functors such as  $\mathrm{Ext}^{*,*}$  or  $\mathrm{Tor}_{*,*}$ . The goal of this paper is to introduce the notion of higher-order derived functors, and explain how they determine the  $E^r$ -term of these spectral sequences.

**0.1. Theories and models.** Recall that a *theory*, in the sense of Lawvere and Ehresmann, is a small category  $\Theta$  with finite products, and a *model* (or algebra) of the theory is a product-preserving functor  $A : \Theta \rightarrow \mathcal{S}et$ , with natural transformations as model morphisms (see [Bor, §4.1]). Examples of categories of such models include varieties of universal algebras, such as groups, rings, and modules over rings.

In topology such theories often arise by choosing a collection  $\mathcal{A}$  of objects in a model category  $\mathcal{C}$ , and letting  $\Theta_{\mathcal{A}}$  be the full subcategory of the homotopy category  $\mathrm{ho}\mathcal{C}$  consisting of finite products of objects in  $\mathcal{A}$ . When each  $A \in \mathcal{A}$  is a homotopy group object in  $\mathcal{C}$ ,  $\Theta_{\mathcal{A}}$ -models have a natural underlying group structure.

For instance, let  $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^{\infty}$  be the collection of  $\mathbb{F}_p$ -Eilenberg-Mac Lane spaces (or spectra) in the category of topological spaces (respectively, spectra). In this case,  $\Theta_{\mathcal{A}}$ -models are unstable algebras over the mod  $p$  Steenrod algebra  $\mathcal{A}_p$  (respectively, modules over  $\mathcal{A}_p$ ).

**0.2. Topologically enriched theories and models.** Most model categories  $\mathcal{C}$  are topologically enriched – that is, for each  $X, Y \in \mathcal{C}$ , there is a (simplicial or topological) mapping space  $\mathrm{map}_{\mathcal{C}}(X, Y)$ , with continuous compositions, such that  $[X, Y]_{\mathrm{ho}\mathcal{C}}$  is equal to its set of components  $\pi_0 \mathrm{map}_{\mathcal{C}}(X, Y)$  (cf. [DK2]). Therefore, we actually have a much richer structure associated to a theory  $\Theta_{\mathcal{A}}$  as above, given by the full topological subcategory  $\Theta_{\mathcal{A}}$  of  $\mathcal{C}$ , with the same objects as  $\Theta_{\mathcal{A}}$ . This  $\Theta_{\mathcal{A}}$  is a topologically enriched theory, and its models are also topologically enriched functors  $\mathfrak{X} : B \mapsto \mathfrak{X}\{B\}$ : thus  $\mathfrak{X}\{B\}$  is a topological space (or simplicial set), for each  $B \in \Theta_{\mathcal{A}}$ , with a continuous action of each mapping space  $\mathrm{map}(B, B')$  of  $\Theta_{\mathcal{A}}$  on it. In particular, for each  $Y \in \mathcal{C}$  we have a *realizable*  $\Theta_{\mathcal{A}}$ -model  $\mathfrak{M}_{\mathcal{A}}Y$ , defined by  $B \mapsto \mathrm{map}_{\mathcal{C}}(Y, B)$ .

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Applying  $\pi_0$  to each mapping space in  $\Theta_{\mathcal{A}}$  and to each space  $\mathfrak{X}\{B\}$  yields the corresponding “algebraic”  $\Theta_{\mathcal{A}}$ -model: for example, if  $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^{\infty}$  and  $\mathfrak{X} = \mathfrak{M}_{\mathcal{A}}Y$  for some space  $Y$ , we obtain the  $\Theta_{\mathcal{A}}$ -model  $H^*(Y; \mathbb{F}_p)$  (as an unstable algebra over the Steenrod algebra – cf. [Sc]).

**0.3. Postnikov approximations.** Note that given a topological  $\Theta_{\mathcal{A}}$ -model  $\mathfrak{X}$  as above, by applying the Postnikov tower functor  $\{P^n\}_{n=0}^{\infty}$  to each of the mapping spaces in  $\Theta_{\mathcal{A}}$ , and to each space  $\mathfrak{X}\{B\}$ , we obtain a sequence of interpolations between the algebraic  $\Theta_{\mathcal{A}}$ -model and the topological  $\Theta_{\mathcal{A}}$ -model. Moreover, since  $P^n$  preserves products,  $P^n\mathfrak{X}$  is again a topologically enriched model for the topological theory  $P^n\Theta_{\mathcal{A}}$ , with both in fact enriched in the category  $\mathcal{T}^{(n)}$  of  $n$ -types.

Furthermore, given an “algebraic” category  $\mathcal{D}_n$  and a product-preserving functor  $\gamma_n : \mathcal{T}^{(n)} \rightarrow \mathcal{D}_n$  which is an equivalence on homotopy categories – that is, an “algebraic model” for  $n$ -types (see [Br, CG, ELS, HP, L, W]) –  $\gamma_n$  provides an “algebraic” version of any  $P^n\Theta_{\mathcal{A}}$ -model  $P^n\mathfrak{X}$ .

For example, for  $n = 0$  we can take  $\mathcal{D}_0$  to be the category of sets, with  $\gamma_0$  given by  $\pi_0$ . For  $n = 1$ , we can take  $\mathcal{D}_1$  to be the category of groupoids, and  $\gamma_1$  is then the fundamental groupoid functor  $\hat{\pi}_1$ . For  $n = 2$ , there are various categories of 2-groupoids which model 2-types; in particular, the version of 2-groupoids constructed in [BP1] has such a product-preserving functor  $\gamma_2$ . See [Ba3, BP2] for the general case.

**0.4. Mod  $p$  Eilenberg-Mac Lane spaces.** Two applications of these notions are provided by two results for a simply-connected space or connective spectrum  $Y$  and the mod  $p$  Eilenberg-Mac Lane theory  $\Theta_{\mathcal{A}}$  (with  $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^{\infty}$ ):

**Theorem A.** *Up to  $p$ -equivalence, one can recover  $Y$  functorially from the  $\Theta_{\mathcal{A}}$ -model  $\mathfrak{M}_{\mathcal{A}}Y$ .*

[See Theorem 3.18 below].

**Theorem B.** *The  $P^n\Theta_{\mathcal{A}}$ -model  $P^n\mathfrak{M}_{\mathcal{A}}Y$  determines the Adams spectral sequence converging to  $\pi_*Y$  up the  $E^{n+2}$ -term.*

[See Theorem 7.4 below].

Hence an algebraic version of the  $P^n\Theta_{\mathcal{A}}$ -model  $P^n\mathfrak{M}_{\mathcal{A}}Y$  provides a tool for computing successive terms in the Adams spectral sequence:

For spectra, in the case  $n = 0$ , Adams’ original construction in [Ad] identifies the  $E^2$ -term of the Adams spectral sequence as a derived functor  $\text{Ext}_{\mathcal{A}_p}^{**}$ , and this was extended in [BK2] to the unstable case. Similarly, for  $n = 1$ , the first author’s work with Jibladze in [BJ1] described the differential  $d_2 : E_2 \rightarrow E_2$  in terms of a certain secondary  $\text{Ext}$ .

This motivates us to introduce the notion of *higher order derived functors*, which make Theorem B precise by providing a way to express the differentials in the Adams spectral sequence in terms of an (arbitrary) resolution of  $P^n\mathfrak{M}_{\mathcal{A}}Y$ .

**0.5. Remark.** This paper is formulated in terms of a simplicial enrichment, but one could work instead with topological function spaces. Similarly, we discuss only unstable (simplicial or cosimplicial) spectral sequences here, but our results apply also to their stable versions.

## 1. MAPPING ALGEBRAS

In the cases of interest to us, simplicially (or topologically) enriched theories and their models have some additional features, which we codify as follows (cf. [BB2, §8]):

**1.1. Definition.** Let  $\mathcal{C}$  be a pointed simplicial model category, and  $\mathcal{A}$  a collection of (homotopy) group objects in  $\mathcal{C}$ . As in §0.2, we let  $\Theta_{\mathcal{A}}$  denote the full sub-simplicial category of  $\mathcal{C}$  containing  $\mathcal{A}$  and closed under finite products and loops. We assume that all objects in  $\Theta_{\mathcal{A}}$  are fibrant and cofibrant. A functor  $\mathfrak{X} : \Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$  into the category of pointed simplicial sets is called an  $\mathcal{A}$ -mapping algebra if it preserves all limits inside  $\Theta_{\mathcal{A}}$ , including products and loops. This means if we denote the value of  $\mathfrak{X}$  at  $B \in \Theta_{\mathcal{A}}$  by  $\mathfrak{X}\{B\} \in \mathcal{S}_*$ , the natural maps:

$$(1.2) \quad \mathfrak{X}\left\{\prod_{i=1}^N B_i\right\} \rightarrow \prod_{i=1}^N \mathfrak{X}\{B_i\} \quad \text{and} \quad \mathfrak{X}\{\Omega B\} \rightarrow \Omega \mathfrak{X}\{B\}$$

are isomorphisms for all  $B, B_i \in \Theta_{\mathcal{A}}$ , where  $\Omega$  denotes the loop functor both in  $\mathcal{C}$  and in  $\mathcal{S}_*$  (see [Q, I, §2]). Similarly for any other limits in  $\mathcal{C}$ , where both the objects and the cone happen to be in  $\Theta_{\mathcal{A}}$ . The category of  $\mathcal{A}$ -mapping algebras is denoted by  $\mathcal{M}_{\mathcal{A}}$ .

Since this is a full subcategory of the category  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  of all simplicial functors  $\Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$ , the category  $\mathcal{M}_{\mathcal{A}}$  is simplicially enriched (cf. [DK1, §1]).

For  $\mathcal{A} \subseteq \mathcal{C}$  as above, for any fixed  $Y \in \mathcal{C}$  we obtain a *realizable*  $\mathcal{A}$ -mapping algebra  $\mathfrak{M}_{\mathcal{A}}Y$  by setting  $(\mathfrak{M}_{\mathcal{A}}X)\{B\} := \text{map}_{\mathcal{C}}(B, X)$  for each  $B \in \Theta_{\mathcal{A}}$ . This defines a functor  $\mathfrak{M}_{\mathcal{A}} : \mathcal{C} \rightarrow \mathcal{M}_{\mathcal{A}}$ .

In particular, an  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$  is *free* if it is an arbitrary coproduct (in  $\mathcal{M}_{\mathcal{A}}$ ) of  $\mathcal{A}$ -mapping algebras of the form  $\mathfrak{M}_{\mathcal{A}}B_{\alpha}$  for  $B_{\alpha} \in \Theta_{\mathcal{A}}$ . Such free  $\mathcal{A}$ -mapping algebras are (homotopy) cogroup objects in  $\mathcal{M}_{\mathcal{A}}$  – and hence in the simplicial category  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  of all simplicial functors  $\Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$  (compare [BB2, Lemma 8.13]).

**1.3. Example.** For  $\mathcal{C} = \mathcal{S}_*$ , let  $\mathcal{A} := \{K(R, n)\}_{n=1}^{\infty}$  for some commutative ring  $R$ . Then  $\Theta_{\mathcal{A}}$  is the simplicially enriched category of  $R$ -GEMs (that is, generalized Eilenberg-Mac Lane spaces of free  $R$ -modules) of finite type. The resulting mapping algebras are called  *$R$ -mapping algebras*.

**1.4. The model category.** If  $\Theta_{\mathcal{A}}$  is a topologically enriched theory as above, the category  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  of all simplicial functors has model category structure in which weak equivalences and fibrations are defined objectwise (cf. [DK1, §7], and compare [BB2, §8]).

Denote by  $\mathcal{M}_{\mathcal{A}}^{\text{free}} \subseteq \mathcal{S}_*^{\Theta_{\mathcal{A}}}$  the full subcategory of all functors weakly equivalent to free  $\mathcal{A}$ -mapping algebras.

**1.5. Definition.** For any simplicial category  $\mathcal{D}$ , let  $P^n\mathcal{D}$  denote the simplicial category (with the same objects as  $\mathcal{D}$ ) obtained from  $\mathcal{D}$  by applying the  $n$ -th Postnikov section functor  $P^n$  to each mapping space in  $\mathcal{D}$ .

In particular, given  $\mathcal{A} \subseteq \mathcal{C}$  as in §1.1, we can apply  $P^n$  to any  $\Theta_{\mathcal{A}}$ -model  $\mathfrak{X}$  to obtain the corresponding  $P^n\Theta_{\mathcal{A}}$ -model  $P^n\mathfrak{X}$  (§0.3), and define an  *$n$ - $\mathcal{A}$ -mapping algebra* to be a (product-preserving)  $P^n\Theta_{\mathcal{A}}$ -model  $\mathfrak{V} : P^n\Theta_{\mathcal{A}} \rightarrow P^n\mathcal{S}_*$  such that the natural map  $P^{n-1}\mathfrak{V}\{\Omega B\} \rightarrow \Omega \mathfrak{V}\{B\}$  is an isomorphism for every  $B \in \Theta_{\mathcal{A}}$ . The category of  $n$ - $\mathcal{A}$ -mapping algebras will be denoted by  $n\text{-}\mathcal{M}_{\mathcal{A}}$ , and the  $n$ -th Postnikov section functor extends to a functor  $P^n : \mathcal{M}_{\mathcal{A}} \rightarrow n\text{-}\mathcal{M}_{\mathcal{A}}$ .

**1.6. Postnikov stems.** In [BB1, §1], we introduced the notion of the *Postnikov  $n$ -stem*  $\mathcal{P}[n]Y$  of a space  $Y \in \mathcal{S}_*$ : that is, the collection of  $(k-1)$ -connected  $(n+k+1)$ -coconnected  *$n$ -windows*  $\{P^{n+k}X\langle k-1 \rangle\}_{k=1}^{\infty}$ , equipped with the composite

$$P^{n+k+1}X\langle k+1 \rangle \rightarrow P^{n+k}X\langle k+1 \rangle \rightarrow P^{n+k}X\langle k \rangle$$

(Postnikov fibration and covering map) as structure maps.

One can apply this functor to the mapping spaces of a restricted simplicial object  $X_\bullet$  in  $\mathcal{S}_*$  (that is, only considering  $\text{map}_*(X_m, X_k)$  for  $m > k$ ) to obtain its *simplicial Postnikov  $n$ -stem*  $\mathcal{P}[n]X_\bullet$ ; by definition its structure includes the simplicial composition maps with  $P^n \text{map}_*(\mathbf{S}^r, X_m)$  for all  $m \geq 0$  and  $r \geq 1$ . Similarly for a (restricted) cosimplicial space  $W^\bullet$ . Denote the category of simplicial Postnikov  $n$ -stems by  $\text{Stem}_{\Delta_{\text{op}}}^n$ , and that of cosimplicial Postnikov  $n$ -stems by  $\text{Stem}_{\Delta}^n$ .

We then showed in [BB1, Theorem 3.14 & Corollary 3.17]:

**1.7. Theorem.** *The homotopy spectral sequence of a (Reedy fibrant) simplicial space  $X_\bullet$  up to the  $E^{n+2}$ -term depends only on its simplicial Postnikov  $n$ -stem  $\mathcal{P}[n]X_\bullet$ . Similarly, the homotopy spectral sequence of a (fibrant) cosimplicial space  $W^\bullet$  up to the  $E_{n+2}$ -term depends only on its cosimplicial Postnikov  $n$ -stem  $\mathcal{P}[n]W^\bullet$ .*

For a different proof, see Lemmas 5.16 and 5.8 below.

**1.8. Extended theories and models.** We defined our enriched theory  $\Theta_{\mathcal{A}}$  to consist of *finite* products of the form  $\prod_{i=1}^N A_i$  ( $A_i \in \mathcal{A}$ ) in order to ensure that it is a small category. However, for our applications we need to extend our models  $\mathfrak{X} : \Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$  to the full simplicial subcategory  $\mathcal{C}_{\mathcal{A}}$  in  $\mathcal{C}$ , whose objects are *arbitrary* products (and loops) of objects from  $\mathcal{A}$ . Unfortunately,  $\mathcal{C}_{\mathcal{A}}$  is no longer small, so we do not have a good model category structure on  $\mathcal{S}_*^{\mathcal{C}_{\mathcal{A}}}$ .

However, when  $\mathcal{A} = \{K(R, n)\}_{n=1}^\infty$  and  $R$  is a field (or a compact abelian group), we can extend our notions of  $\Theta_{\mathcal{A}}$ -models and  $\mathcal{A}$ -mapping algebras as follows: we construct a new simplicially enriched category  $\hat{\mathcal{C}}_{\mathcal{A}}$ , with the same objects as  $\mathcal{C}_{\mathcal{A}}$ , by requiring that  $\text{map}_{\hat{\mathcal{C}}_{\mathcal{A}}}(B, -)$  commute with products in the target, and setting

$$(1.9) \quad \text{map}_{\hat{\mathcal{C}}_{\mathcal{A}}}(B, C) := \lim_{\vec{i}} \text{map}(B_{\vec{i}}, C),$$

where  $\vec{i} = \{i_1, \dots, i_k\}$  runs over all finite subsets of indices of  $I$ , so  $B_{\vec{i}} := \prod_{j=1}^k K(R, n_{i_j})$  runs over all finite sub-products of  $B$ . In other words, infinite products in  $\mathcal{C}_{\mathcal{A}}$  become (enriched) weak products in  $\hat{\mathcal{C}}_{\mathcal{A}}$ .

Recall that for any  $X \in \mathcal{S}_*$  there is a natural weak equivalence  $\text{map}(X, K(R, n)) \rightarrow \lim_i \text{map}(X_i, K(R, n))$  where  $(X_i)_{i \in I}$  is the directed system of finite subcomplexes of  $X$  (see [EiS, VIII (F)] and [HM, §2]). This implies that the natural functor of simplicial categories  $\mathcal{C}_{\mathcal{A}} \rightarrow \hat{\mathcal{C}}_{\mathcal{A}}$  is an objectwise weak equivalence. Thus any  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$ , represented by  $\Theta_{\mathcal{A}}$ , yields an *extended*  $\mathcal{A}$ -mapping algebra  $\hat{\mathfrak{X}}$ , represented by  $\hat{\mathcal{C}}_{\mathcal{A}}$ , and required to preserve not only the specified loops, finite products, and other strict limits inside  $\hat{\mathcal{C}}_{\mathcal{A}}$ , but takes infinite products to weak products in (1.9). The category of extended  $\mathcal{A}$ -mapping algebras will be denoted by  $\hat{\mathcal{M}}_{\mathcal{A}}$ . Note that  $\hat{\mathfrak{M}}_{\mathcal{A}} B$  is in  $\hat{\mathcal{M}}_{\mathcal{A}}^{\text{free}}$  for any  $B \in \mathcal{C}_{\mathcal{A}}$ .

In order to work with a small category, we can restrict to the full-subcategory  $\mathcal{C}_{\mathcal{A}}^\lambda$  of  $\mathcal{C}_{\mathcal{A}}$  (or  $\hat{\mathcal{C}}_{\mathcal{A}}$ ), in which we only take products of cardinality  $\leq \lambda$  for some limit cardinal  $\lambda$ .

Dually, if  $\mathcal{A}$  is a collection of cogroup objects, we denote by  $\mathcal{C}^{\hat{\mathcal{A}}}$  the full simplicial subcategory of  $\mathcal{C}$ , whose objects are *arbitrary* coproducts (and suspensions) of objects from  $\mathcal{A}$ .

**1.10. Remark.** Our definition of  $\mathcal{A}$ -mapping algebras depends on the specific choice of  $\mathcal{A} \subseteq \mathcal{C}$ , and replacing it by a collection  $\mathcal{A}'$  of weakly equivalent objects yields a different notion of  $\mathcal{A}'$ -mapping algebra (though the categories  $\mathcal{C}_{\mathcal{A}}^\lambda$  and  $\mathcal{C}_{\mathcal{A}'}^\lambda$  are  $(\mathcal{S}, \mathcal{O})$ -equivalent (cf. [DK1, §1]), where  $\mathcal{O} := \text{Obj}(\mathcal{C}_{\mathcal{A}}^\lambda)$ ).

We may remedy this by using a lax version of  $\mathcal{C}_A^\lambda$ : that is, replacing the  $(\mathcal{S}, \mathcal{O})$ -category  $\mathcal{C}_A^\lambda$  by a cofibrant and fibrant version  $\tilde{\mathcal{C}}_A^\lambda$ . For example, we may take the free category on  $(\mathcal{C}_A^\lambda)_0$  as the 0-simplices of  $\tilde{\mathcal{C}}_A^\lambda$ , and so on (see [BBD, §6.3]). A *lax  $\mathcal{A}$ -mapping algebra* is then a (strict) simplicial functor  $\tilde{\mathfrak{X}} : \tilde{\mathcal{C}}_A^\lambda \rightarrow \mathcal{S}_*$ , and if  $A'$  is weakly equivalent to  $A$ , any lax  $A'$ -mapping algebra is weakly equivalent to a lax  $\mathcal{A}$ -mapping algebra.

When  $\mathcal{A} = \{K(R, n)\}_{n=1}^\infty$  as in §1.8, we do not need to restrict to  $\mathcal{C}_A^\lambda$ , since lax  $\mathcal{A}$ -mapping algebras always respect (1.9), up to homotopy.

**1.11. The dual version.** The notion of an  $\mathcal{A}$ -mapping algebra as defined above dualizes in the obvious way: starting with a collection  $\hat{\mathcal{A}}$  of homotopy cogroup objects in  $\mathcal{C}$ , the opposite  $\Theta^{\hat{\mathcal{A}}}$  of the sub-simplicial category of  $\mathcal{C}$  whose objects are generated by  $\hat{\mathcal{A}}$  under suspensions and finite coproducts (and again are fibrant and cofibrant) is again a simplicially enriched theory, whose models are *dual  $\hat{\mathcal{A}}$ -mapping algebras*. One can define a realizable dual  $\hat{\mathcal{A}}$ -mapping algebra  $\mathfrak{M}^{\hat{\mathcal{A}}}Y$ , free dual  $\hat{\mathcal{A}}$ -mapping algebras, as well as  $n$ -dual  $\hat{\mathcal{A}}$ -mapping algebras, as before, and the free dual  $\hat{\mathcal{A}}$ -mapping algebras are again homotopy cogroup objects in  $\mathcal{S}_*^{\Theta^{\hat{\mathcal{A}}}}$ .

Typically,  $\hat{\mathcal{A}}$  will be generated under suspensions by a single cofibrant homotopy cogroup object  $A$  in  $\mathcal{C}$ ; the motivating example is the collection of spheres  $\{S^n\}_{n=1}^\infty$  in  $\mathcal{T}_*$ . The crucial observation is that in this case a realizable dual  $\hat{\mathcal{A}}$ -mapping algebra  $\mathfrak{M}^{\hat{\mathcal{A}}}Y$  is simply a loop space  $\mathfrak{X}\{S^1\} = \Omega Y$ , together with its iterated loops and products thereof, and the action of the free  $\mathcal{A}$ -mapping algebra  $\mathfrak{M}^{\hat{\mathcal{A}}}S^1$  on  $\Omega Y$  and its products encodes its  $A_\infty$ -structure. Thus one can functorially recover  $Y$  from  $\mathfrak{M}^{\hat{\mathcal{A}}}Y$  in this situation, using [Ma2].

Note that for  $\hat{\mathcal{A}} = \{S^n\}_{n=1}^\infty$ , the  $n$ -dual  $\hat{\mathcal{A}}$ -mapping algebra  $P^n\mathfrak{M}^{\hat{\mathcal{A}}}X$  is equivalent to the Postnikov stem of  $X$  (see §1.6), so for ordinary  $\mathcal{A}$ -mapping algebras we can think of the  $n$ - $\mathcal{A}$ -mapping algebra  $P^n\mathfrak{M}_\mathcal{A}X$  as the  $\mathcal{A}$ - $n$ -stem of  $X$ .

## 2. HIGHER ORDER DERIVED FUNCTORS

In order to define the notion of higher order derived functors, we first recall a construction in homotopical algebra:

**2.1. The resolution model category structure.** Let  $\mathcal{C}$  be a pointed, cofibrantly generated, simplicial, and right proper model category, equipped with a set  $\hat{\mathcal{A}}$  of homotopy cogroup objects in  $\mathcal{C}$ . There is a *resolution* model category structure on the category  $s\mathcal{C}$  of simplicial objects over  $\mathcal{C}$  defined as follows (based on the treatment of simplicial spaces in [DKSt1] – cf. [J]):

For any  $Y \in \mathcal{C}$ , let  $\pi^{\hat{\mathcal{A}}}Y$  denote the  $(\hat{\mathcal{A}} \times \mathbb{N})$ -graded group  $(\pi_* \text{map}_\mathcal{C}(A, Y))_{A \in \hat{\mathcal{A}}}$ , and define a simplicial map  $f : X_\bullet \rightarrow Y_\bullet$  to be:

- (i) a *weak equivalence* if  $\pi^{\hat{\mathcal{A}}}f$  is a weak equivalence of  $(\mathcal{A} \times \mathbb{N})$ -graded simplicial groups.
- (ii) a *cofibration* if it is (a retract of) a homotopically  $\mathcal{A}$ -free map – that is for each  $n \geq 0$ , there is a cofibrant object  $W_n$  in  $\mathcal{C}$  which is weakly equivalent to a coproduct of objects from  $\mathcal{A}$ , and a map  $\varphi_n : W_n \rightarrow Y_n$  in  $\mathcal{C}$  inducing a trivial cofibration  $(X_n \amalg_{L_n X_\bullet} L_n Y_\bullet) \amalg W_n \rightarrow Y_n$ . Here  $L_n Y_\bullet$  is the  $n$ -th latching object for  $Y_\bullet$ .
- (iii) a *fibration* if it is a Reedy fibration (cf. [Hir, 15.3]) and  $\pi^{\hat{\mathcal{A}}}f$  is a fibration of  $(\mathcal{A} \times \mathbb{N})$ -graded simplicial groups.

The dual case, in which  $\mathcal{A}$  be is a collection of homotopy abelian group objects in  $\mathcal{C}$ , yielding a resolution model category structure on the category  $\mathcal{C}$  of cosimplicial objects over  $\mathcal{C}$ , is treated in detail in [Bou].

**2.2. Simplicial  $\mathcal{A}$ -mapping algebras and dual  $\hat{\mathcal{A}}$ -mapping algebras.** Even when  $\mathcal{C} = \mathcal{S}_*$  or  $\mathcal{T}_*$  and  $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^\infty$ , there seems to be no useful model category structure on the category  $\mathcal{M}_{\mathcal{A}}$  of  $\mathcal{A}$ -mapping algebras. However, we do have an appropriate model category structure on the functor category  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  (§1.4), and any free  $\mathcal{A}$ -mapping algebra is in fact a (homotopy) cogroup object for  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$ . Thus if we let  $\hat{\mathcal{A}} := \{\mathfrak{M}_{\mathcal{A}} A\}_{A \in \mathcal{A}}$ , we obtain a resolution model category structure on the category  $s\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  of *simplicial* simplicially-enriched functors  $\Theta_{\mathcal{A}} \rightarrow \mathcal{S}_*$ .

Moreover, the category  $(P^n \mathcal{S}_*, \mathcal{O})\text{-Cat}$  of categories enriched in pointed  $n$ -types (cf. §1.5) again forms a model category, and the free  $n$ - $\mathcal{A}$ -mapping algebras  $P^n \mathfrak{M}_{\mathcal{A}} A$  are again (homotopy) cogroup objects for the category  $P^n \mathcal{S}_*^{P^n \Theta_{\mathcal{A}}}$  of  $(P^n \mathcal{S}_*, \mathcal{O})$ -functors. Thus we also have a resolution model category structure on  $s(P^n \mathcal{S}_*^{P^n \Theta_{\mathcal{A}}})$ .

Similarly, if  $\hat{\mathcal{A}}$  is a collection of homotopy cogroup objects in a pointed model category  $\mathcal{C}$ , we have the (dual) model category structure on the functor category  $\mathcal{S}_*^{\Theta_{\hat{\mathcal{A}}}}$ . Again  $\hat{\mathcal{A}} := \{\mathfrak{M}_{\hat{\mathcal{A}}} A\}_{A \in \hat{\mathcal{A}}}$  is a collection of homotopy cogroup objects in  $\mathcal{S}_*^{\Theta_{\hat{\mathcal{A}}}}$ , so we have a resolution model category structure on the category  $s(\mathcal{S}_*^{\Theta_{\hat{\mathcal{A}}}})$  of simplicial simplicially-enriched functors, as well as on  $s(P^n \mathcal{S}_*^{P^n \Theta_{\hat{\mathcal{A}}}})$ .

By definition of the relevant resolution model category structures we see:

**2.3. Lemma.** *If  $\mathcal{A}$  is a collection of homotopy group objects in  $\mathcal{C}$ , and  $Y \rightarrow W^\bullet$  is a cosimplicial  $\mathcal{A}$ -resolution of  $Y \in \mathcal{C}$  in  $c\mathcal{C}$ , then  $\mathfrak{M}_{\mathcal{A}} W^\bullet$  is a free simplicial resolution of  $\mathfrak{M}_{\mathcal{A}} Y$  in  $s\mathcal{S}_*^{\Theta_{\mathcal{A}}}$ . Similarly, If  $\hat{\mathcal{A}}$  is a collection of homotopy cogroup objects in  $\mathcal{C}$ , and  $W_\bullet \rightarrow Y$  is a simplicial  $\hat{\mathcal{A}}$ -resolution of  $Y$  in  $s\mathcal{C}$ , then  $\mathfrak{M}_{\hat{\mathcal{A}}} W_\bullet$  is a free simplicial resolution of  $\mathfrak{M}_{\hat{\mathcal{A}}} Y$  in  $s\mathcal{S}_*^{\Theta_{\hat{\mathcal{A}}}}$ .*

**2.4. Definition.** Let  $\mathcal{A}$  be a collection of homotopy group objects in the pointed simplicial model category  $\mathcal{C}$ , and let  $F : \mathcal{M}_{\mathcal{A}}^{\text{free}} \rightarrow \mathcal{S}_*$  (§1.4) be an enriched functor, which can be prolonged to  $\tilde{F} : s\mathcal{M}_{\mathcal{A}}^{\text{free}} \rightarrow c\mathcal{S}_*$ . If  $\tilde{F}$  preserves weak equivalences (in the above model category  $s(\mathcal{S}_*^{\Theta_{\mathcal{A}}})$ ), we call  $F$  an  $\mathcal{A}$ -homotopy functor. This will hold, in particular, if  $F$  itself preserves all weak equivalences.

The total  $n$ -th order left derived functor  $\mathbb{L}^n F$  of such an  $F$ , applied to  $Y \in \mathcal{C}$ , is defined to be the simplicial  $n$ -stem  $(\mathbb{L}^n F)Y := \mathcal{P}[n] \tilde{F} \mathfrak{V}_\bullet \in \text{Stem}_{\Delta}^n$  (cf. §1.6), where  $\mathfrak{V}_\bullet$  is a resolution of  $\mathfrak{M}_{\mathcal{A}} Y$  in the  $\mathcal{A}$ -resolution model category structure on  $s(\mathcal{S}_*^{\Theta_{\mathcal{A}}})$ .

If  $F : (\mathcal{M}_{\mathcal{A}}^{\text{free}})^{\text{op}} \rightarrow \mathcal{S}_*$  is contravariant, its total  $n$ -th order left derived functor  $(\mathbb{L}^n F)Y$  is defined to be the cosimplicial  $n$ -stem  $\mathcal{P}[n] \tilde{F} \mathfrak{V}_\bullet \in \text{Stem}_{\Delta}^n$ .

**2.5. Remark.** Note that even when the collection  $\hat{\mathcal{A}}$  consists of homotopy cogroup objects in  $\mathcal{C}$ , so we replace  $\mathcal{A}$ -mapping algebras by dual  $\hat{\mathcal{A}}$ -mapping algebras, the free dual  $\hat{\mathcal{A}}$ -mapping algebras are still homotopy cogroup objects in  $\mathcal{S}_*^{\Theta_{\hat{\mathcal{A}}}}$ , so we still work in the model category of *simplicial* dual  $\hat{\mathcal{A}}$ -mapping algebras.

Note also that as for any simplicially enriched category,  $\text{Stem}_{\Delta}^n$  and  $\text{Stem}_{\Delta}^n$  have a notion of Dwyer-Kan equivalence (i.e., objectwise weak equivalence) – cf. [DK3, §1].

**2.6. Proposition.** *If  $F : \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{S}_*$  is an  $\mathcal{A}$ -homotopy functor, the total  $n$ -th order left derived functor  $\mathbb{L}^n F$  is well defined up to weak equivalence in the resolution model category structure on  $s\mathcal{M}_{\mathcal{A}}$ , and depends on  $Y \in \mathcal{C}$  only up to  $\mathcal{A}$ -equivalence.*

*Proof.* Any two  $\mathcal{A}$ -resolutions  $\mathfrak{V}_\bullet$  and  $\mathfrak{W}_\bullet$  of  $\mathfrak{M}_\mathcal{A}Y$  are homotopy equivalent in  $s(\mathcal{S}_*^{\Theta_\mathcal{A}})$ , since they are both fibrant and cofibrant, so there are maps of simplicial  $\mathcal{A}$ -mapping algebras  $\mathfrak{f} : \mathfrak{V}_\bullet \rightarrow \mathfrak{W}_\bullet$  and  $\mathfrak{g} : \mathfrak{W}_\bullet \rightarrow \mathfrak{V}_\bullet$  with simplicial homotopies  $\mathfrak{G} : \mathfrak{g} \circ \mathfrak{f} \rightarrow \text{Id}_{\mathfrak{W}_\bullet}$  and  $\mathfrak{F} : \mathfrak{f} \circ \mathfrak{g} \rightarrow \text{Id}_{\mathfrak{V}_\bullet}$  as in [Ma1, I, §5]. Since the functor  $P^n \mathfrak{M}_\mathcal{B}(F-)$  is prolonged from  $\mathcal{M}_\mathcal{A}$ , applying it to  $\mathfrak{f}$  and  $\mathfrak{g}$  shows that  $P^n(F\mathfrak{V}_\bullet)$  is homotopy equivalent to  $P^n(F\mathfrak{W}_\bullet)$  so in particular it is weakly equivalent in the  $E^2$ -model category structure on (co)simplicial spaces (cf. [DKSt1]).

If  $Y$  and  $Y'$  are  $\mathcal{A}$ -equivalent in  $\mathcal{C}$ , then by definition  $\mathfrak{M}_\mathcal{A}Y$  and  $\mathfrak{M}_\mathcal{A}Y'$  are weakly equivalent in the model category  $\mathcal{S}_*^{\Theta_\mathcal{A}}$ . Therefore, any two  $\mathcal{A}$ -resolutions  $\mathfrak{V}_\bullet \rightarrow \mathfrak{M}_\mathcal{A}Y$  and  $\mathfrak{V}'_\bullet \rightarrow \mathfrak{M}_\mathcal{A}Y'$  are also weakly equivalent in the model category  $s(\mathcal{S}_*^{\Theta_\mathcal{A}})$ , and thus homotopy equivalent, yielding weakly equivalent total derived functors.  $\square$

**2.7. Application to spectral sequences.** A number of spectral sequences – including the unstable Adams spectral sequences of [6A, BK2] and [BCM, §4], Rector’s version of the Eilenberg-Moore spectral sequence (cf. [Re2]), and Anderson’s generalization of the latter (cf. [An]) – are obtained by applying a functor  $G : \mathcal{C} \rightarrow \mathcal{S}_*$  to a suitable (co)simplicial resolution  $Y \rightarrow W^\bullet$  or  $W_\bullet \rightarrow Y$ , and then using the homotopy spectral sequence for the (co)simplicial space  $GW^\bullet$  or  $GW_\bullet$  to obtain information about its  $\pi_* \text{Tot } GW^\bullet$  (or  $\pi_* \|GW_\bullet\|$ ), which is usually related to  $GY$ .

Thus in order to apply our notion of higher order derived functors to spectral sequence calculations, we must replace the given functor  $G : \mathcal{C} \rightarrow \mathcal{S}_*$  by a  $\mathcal{A}$ -homotopy functor  $F : \mathcal{M}_\mathcal{A} \rightarrow \mathcal{S}_*$ . More precisely, we can reformulate Theorem 1.7 in the following form:

**2.8. Theorem.** *Let  $\mathcal{A}$  be a collection of homotopy group objects in a pointed simplicial model category  $\mathcal{C}$  as above,  $F : \mathcal{M}_\mathcal{A}^{\text{free}} \rightarrow \mathcal{S}_*$  an  $\mathcal{A}$ -homotopy functor and  $G := F \circ \mathfrak{M}_\mathcal{A} : \mathcal{C} \rightarrow \mathcal{S}_*$ . Then for any  $Y \in \mathcal{C}$  and  $\mathcal{A}$ -resolution  $Y \rightarrow W^\bullet$ , the homotopy spectral sequence for  $GW^\bullet$  is determined up to the  $E_{n+1}$ -term by the  $n$ -th order derived functors of  $F$  applied to  $\mathfrak{M}_\mathcal{A}Y$ .*

**2.9. Remark.** There are of course analogous results when  $\hat{\mathcal{A}}$  be a collection of homotopy cogroup objects in  $\mathcal{C}$ , or when  $F$  is contravariant.

In fact, given  $G : \mathcal{C} \rightarrow \mathcal{S}_*$ , we need only show that  $G = F \circ \mathfrak{M}_\mathcal{A}$  when applied to objects in the subcategory  $\mathcal{C}_\mathcal{A}$  (or  $\mathcal{C}_\mathcal{A}^\lambda$ ) of  $\mathcal{C}$  (see §1.8), since by Lemma 2.3 we may assume that the  $\mathcal{A}$ -mapping algebra-resolution  $\mathfrak{V}_\bullet$  of  $\mathfrak{M}_\mathcal{A}Y$  is obtained from a cosimplicial  $\mathcal{A}$ -resolution  $Y \rightarrow W^\bullet$  (or a simplicial  $\hat{\mathcal{A}}$ -resolution  $W_\bullet \rightarrow Y$  by applying  $\mathfrak{M}_\mathcal{A}$  (or  $\mathfrak{M}^{\hat{\mathcal{A}}}$ ). By construction, each  $W^n$  (or  $W_n$ ) is in  $\mathcal{C}_\mathcal{A}$  (or  $\mathcal{C}^{\hat{\mathcal{A}}}$ ) – cf. §2.1. It is not easy to do this, in general – in fact, Section 3 below is dedicated to the example relevant to the Adams spectral sequence.

**2.10. The dual case.** If  $\mathcal{C} = \mathcal{S}_*$  and  $\hat{\mathcal{A}} = \{S^n\}_{n=1}^\infty$ , most common continuous functors  $F : \mathcal{S}_* \rightarrow \mathcal{S}_*$  are  $\hat{\mathcal{A}}$ -homotopy functors when restricted to the subcategory  $\mathcal{S}_0$  of pointed connected spaces, since by §1.11 the functor  $X \mapsto \mathfrak{M}_\mathcal{A}X$  has a homotopy inverse for  $X \in \mathcal{S}_0$ . Note that we only need to apply  $F$  to objects in  $\mathcal{C}^{\hat{\mathcal{A}}}$ , which are all connected. Thus we have many examples to consider in this context:

- (a) The loop functor  $\Omega^k$  clearly factors through  $\mathfrak{M}^{\hat{\mathcal{A}}}$ ,
- (b) More generally, the mapping space functor  $\text{map}_*(E, -)$  factors through  $\mathfrak{M}^{\hat{\mathcal{A}}}$  when  $E$  is a suspension.
- (c) Other examples are various limit functors, such as the Cartesian product  $\times : \mathcal{S}_* \times \mathcal{S}_* \rightarrow \mathcal{S}_*$ .

- (d) The abelianization (free abelian group) functor  $\mathbb{Z} : \mathcal{S}_0 \rightarrow \mathcal{S}_0$  factors through  $\mathfrak{M}^{\hat{\mathcal{A}}}$ .
- (e) The wedge functor  $\vee : \mathcal{S}_* \times \mathcal{S}_* \rightarrow \mathcal{S}_*$  factors through  $\mathfrak{M}^{\hat{\mathcal{A}}}$  when restricted to suspensions, by the Hilton-Milnor Theorem (cf. [Mi]). This suffices in our case, because any  $B$  in  $\mathcal{C}^{\hat{\mathcal{A}}}$  is a suspension.

Segal's homology spectral sequence (cf. [Se]), the van Kampen spectral sequence (cf. [St]), and the Hurewicz spectral sequence (cf. [Bl]) are all obtained by applying these functors to an  $\hat{\mathcal{A}}$ -resolution of spaces, for  $\hat{\mathcal{A}} = \{S^n\}_{n=1}^\infty$  (see [St]).

### 3. RECOVERING A SPACE FROM ITS $R$ -MAPPING ALGEBRA

Let  $R$  be a commutative ring, and  $\mathcal{A} = \{K(R, n)\}_{n=1}^\infty$ . In this section we describe a functorial construction for recovering a (simply connected) space  $Y \in \mathcal{S}_*$  from its  $R$ -mapping algebra  $\mathfrak{M}_{\mathcal{A}} Y$ , up to  $R$ -completion. In particular, this will allow us to apply Theorem 2.8 to the spectral sequence for  $Y$  with respect to the ring  $R$  (cf. [BK2]).

By [BK1, I, §4.5], we may assume that  $R \subseteq \mathbb{Q}$  or  $R = \mathbb{F}_p$ . We use the Bousfield's resolution model category structure on  $c\mathcal{S}_*$ , for the class of injectives  $\mathcal{G} = \mathcal{M}_{\mathcal{A}}$  consisting of  $R$ -GEMs (cf. [Bou, §3]).

**3.1. Definition.** If  $\mathcal{A}$  is any collection of homotopy group objects in a simplicial category  $\mathcal{C}$ , a *discrete  $\mathcal{A}$ -mapping algebra* is a function  $\mathfrak{Y} : \mathcal{A} \rightarrow \mathcal{S}_*$ , (again written  $B \mapsto \mathfrak{Y}\{B\}$ ), equipped with isomorphisms  $\theta_B : \mathfrak{Y}\{\Omega B\} \rightarrow \Omega \mathfrak{Y}\{B\}$  for each  $B \in \mathcal{A}$ . We can of course extend it to a function on  $\text{Obj } \Theta_{\mathcal{A}}$  by requiring it to preserve products. The category of discrete  $\mathcal{A}$ -mapping algebras is denoted by  $\mathcal{M}_{\mathcal{A}}^\delta$ , with the forgetful functor  $U : \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{A}}^\delta$ . We write  $\mathfrak{X}_\delta$  for  $U\mathfrak{X}$  and  $\mathfrak{M}_{\mathcal{A}}^\delta$  for  $U \circ \mathfrak{M}_{\mathcal{A}}$ .

**3.2. Some simplicial constructions.** Recall that if  $K$  is a pointed Kan complex, the *path object*  $PK \in \mathcal{S}_*$  is defined by:  $(PK)_n := \text{Ker}(d_{I^{n+1}} : K(R, n+1) \rightarrow K(R, 0))$ , where  $d_{I^{n+1}} := d_1 d_2 \dots d_{n+1}$ , and the universal fibration  $p : PK \rightarrow K$  is defined by  $d_0^i$  in each simplicial dimension  $i$ , with  $\Omega A := \text{Ker}(p)$ . We denote the *path fibration* functor  $K \mapsto (PK \xrightarrow{p} K)$  by  $\rho : \mathcal{S}_* \rightarrow (\mathcal{S}_*)^T$ , where  $T$  is the single-arrow category  $(0 \rightarrow 1)$ . This extends to a functor  $\rho : \mathcal{M}_{\mathcal{A}}^\delta \rightarrow (\mathcal{M}_{\mathcal{A}}^\delta)^T$  (compare [BB2, §8.4]).

In particular, if  $K \in s\mathcal{G}p$  is a simplicial group, and  $K^e$  is its zero component, then for each  $i \geq 0$  we have two natural short exact sequences of groups with splittings:

$$(3.3) \quad 0 \longrightarrow (PK)_i \xrightarrow{\text{inc}} K(R, i+1) \xrightleftharpoons[s_{I^{i+1}}]{s_{I^{i+1}}} K(R, 0) \longrightarrow 0$$

for  $s_{I^{i+1}} := s_i \dots s_1 s_0$ ,

$$(3.4) \quad 0 \longrightarrow (\Omega K)_i \xrightarrow{\text{inc}} (PK)_i \xrightleftharpoons[p_i]{s_0} K_i^e \longrightarrow 0 ;$$

and one which is not split (for the canonical projection  $\gamma$ ):

$$(3.5) \quad 0 \longrightarrow K_i^e \xrightarrow{\text{inc}} K(R, i) \xrightarrow{\gamma} \pi_0 K \longrightarrow 0 .$$

Our goal is to describe a special construction of  $\mathcal{G}$ -resolutions (compare [St, §2] and [BB2, §8]).



**3.6. Definition.** Let  $\lambda$  be a fixed limit cardinal, and let  $sR\text{-Mod}$  denote the full subcategory of  $\mathcal{S}_*$  consisting of simplicial  $R$ -modules – that is,  $R$ -GEMs. For any  $B \in sR\text{-Mod}$ , the limit  $L_{(I,B)}$  of  $I$  copies of  $PB$ , all mapping by  $p$  to a single copy of  $B$ :

$$(3.7) \quad L_{(I,B)} := \lim \left( \{PB_{(i)}\}_{i \in I} \xrightarrow{p} B \right)$$

will be called an *elementary  $R$ -Stover space*. An  *$R$ -Stover space* is any product of at most  $\lambda$  elementary  $R$ -Stover spaces.

**3.8. Remark.** Note that since all objects and maps in the diagram of (3.7) are in the category  $sR\text{-Mod}$ , all  $R$ -Stover spaces are  $R$ -GEMs.

To explain our choice of  $\lambda$ , for any  $X \in \mathcal{S}_*$  let us define  $\kappa(X)$  to be the cardinal sum  $\sum_{B \in \Theta_{\mathcal{A}}} |\mathfrak{M}_{\mathcal{A}} X \{B\}|$  where  $\mathcal{A} = \{K(R, n)\}_{n=1}^{\infty}$ . Now given a fixed space  $Y \in \mathcal{S}_*$ , we choose the cardinal  $\lambda$  so that:

- (a)  $\lambda > \kappa(Y)$ ;
- (b) If we iterate the construction (3.7) countably many times:  $B_{i+1} := L_{(I_i, B_i)}$ , beginning with an  $R$ -GEM  $B_0$  with  $\kappa(B_0) \leq \kappa(Y)$ , and with  $|I_i| \leq \kappa$  at each stage, then the final  $R$ -Stover object  $B_{\infty}$  has  $\kappa(B_{\infty}) \leq \lambda$ .

Denote by  $\mathcal{C}_{\mathcal{A}'}^{\lambda}$  the full simplicial category of  $R$ -Stover spaces (in  $\mathcal{S}_*$ ) for this  $\lambda$ . Clearly,  $\mathcal{C}_{\mathcal{A}'}^{\lambda}$  is a theory, and the corresponding extended theory  $\hat{\mathcal{C}}_{\mathcal{A}'}^{\lambda}$  (cf. §1.8) is weakly equivalent to the original  $\hat{\mathcal{C}}_{\mathcal{A}}^{\lambda}$  ( $\mathcal{A} = \{K(R, n)\}_{n=1}^{\infty}$ ). In this section we will therefore assume that our  $\mathcal{A}$ -mapping algebras are in fact extended  $\mathcal{A}'$ -mapping algebras. Alternatively, we may work with lax theories, as in §1.10, in which case any (extended)  $\mathcal{A}$ -mapping algebra is also an (extended)  $\mathcal{A}'$ -mapping algebra, and conversely.

**3.9. Definition.** Let  $\mathfrak{Y}$  be a discrete  $\mathcal{A}$ -mapping algebra. For each  $B \in \mathcal{M}_{\mathcal{A}}$  and  $\phi \in \mathfrak{Y}\{B\}_0$ , the *space of nullhomotopies* for  $\phi$  is the pullback  $N^{\phi}$ :

$$\begin{array}{ccc} N^{\phi} & \xrightarrow{\quad \text{PB} \quad} & P\mathfrak{Y}\{B\} \\ \downarrow & & \downarrow p \\ \{\phi\} & \xrightarrow{\quad \text{inc} \quad} & \mathfrak{Y}\{B\} \end{array}$$

in  $\mathcal{S}_*$ . If  $\phi \not\sim 0$ , then  $N^{\phi} = \emptyset$ ,

The *dual Stover construction* on  $\mathfrak{Y}$  is then defined to be the  $R$ -Stover space:

$$(3.10) \quad T\mathfrak{Y} := \prod_{B \in \mathcal{M}_{\mathcal{A}}} \prod_{\phi \in \mathfrak{Y}\{B\}_0} \lim \left( \{PB_{(\Phi)}\}_{\Phi \in N_0^{\phi}} \xrightarrow{\text{proj}} B_{(\phi)} \right)$$

(see [BB2, §9.10] and compare [St, §2]).

This defines a contravariant functor  $T : \mathcal{M}_{\mathcal{A}}^{\delta} \rightarrow \mathcal{S}_*$ , which actually takes values in the category  $sR\text{-Mod}$ .

**3.11. Proposition.** *The functor  $T$  is right adjoint to  $\mathfrak{M}_{\mathcal{A}}^{\delta} : \mathcal{S}_* \rightarrow \mathcal{M}_{\mathcal{A}}^{\delta}$ .*

*Proof.* Given a map of  $\mathcal{A}$ -mapping algebras  $\psi : \mathfrak{M}_{\mathcal{A}} Y \rightarrow \mathfrak{X}$ , its adjoint  $\tilde{\psi} : Y \rightarrow T\mathfrak{X}$  is determined by mapping  $Y$  into the diagram for the limit (3.10) defining  $T\mathfrak{X}$ , sending  $\phi \in \mathfrak{X}\{K(R, n)\}_0$  to  $\psi(\phi) : Y \rightarrow (K(R, n))_{(\phi)}$ , and  $\Phi \in P(\mathfrak{X}\{K(R, n)\})_0$  to  $\psi(\Phi) : Y \rightarrow (PK(R, n))_{(\Phi)}$ .

Conversely, given  $f : Y \rightarrow T\mathfrak{X}$  in  $\mathcal{S}_*$ , we define  $\hat{f} : \mathfrak{M}_{\mathcal{A}}^{\delta} Y \rightarrow \mathfrak{X}$  in  $\mathcal{M}_{\mathcal{A}}^{\delta}$  by induction on the simplicial dimension of the mapping spaces:

- (a) On the 0-simplices, we send  $\phi \in (\mathfrak{M}_A Y)\{K(R, n)\}_0 = \text{map}_{\mathcal{S}_*}(Y, K(R, n))_0$  to the projection  $\text{proj}_{(K(R, n))_{(\phi)}}(f) \in (\mathfrak{M}_A Y)\{K(R, n)\}_0 = \text{map}_{\mathcal{S}_*}(Y, K(R, n))_0$  onto the factor  $(K(R, n))_{(\phi)}$  in (3.10).
- (b) For a 1-simplex of the form  $\Phi \in N^\phi \subseteq (P(\mathfrak{M}_A Y)\{K(R, n)\})_0 \subseteq ((\mathfrak{M}_A Y)\{K(R, n)\})_1$ , we send  $\Phi$  to  $\text{proj}_{(PK(R, n))_{(\Phi)}}(f) \in P(\mathfrak{M}_A Y)\{K(R, n)\}_0 \subseteq (\mathfrak{M}_A Y)\{K(R, n)\}_1$ .

We then use the split extension (3.3) for the simplicial abelian group  $A = \mathfrak{M}_A Y\{K(R, n)\}$  and  $i = 0$  to extend  $\hat{f}$  to all 1-simplices of  $A$ .

- (c) If we have defined  $\hat{f}\{K(R, n)\} : (\mathfrak{M}_A^\delta Y)\{K(R, n)\} \rightarrow \mathfrak{Y}\{K(R, n)\}$  for all  $n \geq 1$  though simplicial dimension  $i$ , the natural identification

$$(\Omega \mathfrak{M}_A^\delta Y)\{K(R, n)\} \xrightarrow{\cong} (\mathfrak{M}_A^\delta Y)\{\Omega K(R, n)\} = (\mathfrak{M}_A^\delta Y)\{K(R, n-1)\}$$

of §1.1 allows us to use the split extension (3.4) for  $A = \mathfrak{M}_A Y\{K(R, n)\}$  to extend  $\hat{f}$  to  $(PA)_{i+1}$  – where (3.5) defines  $A_{i+1}^e \subseteq A_{i+1}$ , and then (3.3) again extends  $\hat{f}$  to all of  $A_{i+1}$ .

Because the splittings of (3.3) and (3.4) are natural in  $A$  (and thus in  $K(R, n)$ ), this determines  $\hat{f}$  uniquely as a map of  $\mathcal{A}$ -mapping algebras.  $\square$

**3.12. Corollary.** *The composite  $\mathfrak{T} := \mathfrak{M}_A^\delta \circ T$  is a comonad on  $\mathcal{M}_A^\delta$ , and  $Q := T \circ \mathfrak{M}_A^\delta$  is a monad on  $\mathcal{S}_*$ .*

**3.13. Proposition.** *For any  $X \in \mathcal{S}_*$ , the coaugmented cosimplicial space  $X \rightarrow W^\bullet$  defined by  $W^n = Q^{n+1}Y$  is a  $\mathcal{G}$ -weak resolution in the  $\mathcal{A}$ -resolution model category  $c\mathcal{S}_*$ , for  $\mathcal{G} = \mathcal{M}_A$  (see [Bou, §6.1]).*

*Proof.* Evidently,  $W^\bullet$  is  $\mathcal{G}$ -injective in each cosimplicial dimension, since (3.10) is an  $R$ -GEM. To see that  $X \rightarrow W^\bullet$  is a  $\mathcal{A}$ -weak equivalence, it suffices to check this after applying  $\mathfrak{M}_A^\delta$  and checking  $\pi_0(\mathfrak{M}_A W^\bullet)\{K(R, n)\}$  for each  $n \geq 1$ . But the simplicial discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}_\bullet := \mathfrak{M}_A^\delta W^\bullet$  is obtained by iterating the comonad  $\mathfrak{T}$  on  $\mathfrak{M}_A^\delta Y$ , so  $\pi_0 \mathfrak{M}_A^\delta Y \rightarrow \pi_0 \mathfrak{M}_A^\delta W^\bullet$  yields a contractible coaugmented cosimplicial abelian group at each  $K(R, n)$  (see [Mc, VII, §6]).  $\square$

**3.14. Definition.** Note that a *realizable* discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{X} := \mathfrak{M}_A^\delta Y$  has an extra piece of structure, coming from the co-augmentation  $\varepsilon_X : X \rightarrow T\mathfrak{X}$  (adjoint to  $\text{Id}_{\mathfrak{X}}$ ): namely, the counit  $\eta_{\mathfrak{X}} = \mathfrak{M}_A \varepsilon_X : \mathfrak{T}\mathfrak{X} \rightarrow \mathfrak{X}$ , which is a splitting for the unit  $\xi : \mathfrak{X} \rightarrow \mathfrak{T}\mathfrak{X}$  for the monad  $\mathfrak{T}$ . Any discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$  equipped with a map  $\eta_{\mathfrak{X}} : \mathfrak{T}\mathfrak{X} \rightarrow \mathfrak{X}$  fitting into a commutative diagram

$$(3.15) \quad \begin{array}{ccc} \mathfrak{T}\mathfrak{T}\mathfrak{X} & \xrightarrow{\mathfrak{M}_A \varepsilon_{T\mathfrak{X}}} & \mathfrak{T}\mathfrak{X} \\ \mathfrak{T}(\eta_{\mathfrak{X}}) \downarrow & & \downarrow \eta_{\mathfrak{X}} \\ \mathfrak{T}\mathfrak{X} & \xrightarrow{\eta_{\mathfrak{X}}} & \mathfrak{X} \end{array}$$

is called a  $\mathfrak{T}$ -algebra (cf. [Bor, §4.1]).

**3.16. Proposition.** *For any  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$ , the discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}_\bullet$  has a natural  $\mathfrak{T}$ -algebra structure  $\eta_{\mathfrak{X}} : \mathfrak{T}\mathfrak{X}_\bullet \rightarrow \mathfrak{X}_\bullet$ .*

The converse is also true: that is, for any  $\mathfrak{T}$ -algebra  $\mathfrak{Y}$ , there is a functorial choice of an  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$  such that  $\mathfrak{X}_\bullet \cong \mathfrak{Y}$ ; but we shall not need this fact.

*Proof.* Under the assumptions of §3.8, since  $C := T\mathfrak{X}_\bullet$  is an  $R$ -Stover space, it is in  $\Theta_A$  (actually, in  $\Theta_{A'}$ , in the notation there). Therefore,  $\mathfrak{M}_A T\mathfrak{X}_\bullet = \mathfrak{M}_A C$

is a free  $\mathcal{A}$ -mapping algebra, so by the enriched Yoneda Lemma we can define an  $\mathcal{A}$ -mapping algebra-map  $\eta : \mathfrak{M}_{\mathcal{A}}C \rightarrow \mathfrak{X}$  by specifying a 0-simplex  $e$  in  $\mathfrak{X}\{C\}_0$ . By (3.10),  $C$  is a product over  $B \in \mathcal{M}_{\mathcal{A}}$  and  $\phi \in \mathfrak{X}\{B\}_0$  of elementary Stover spaces  $C^{(B,\phi)} := L_{(N_0^\phi, B)}$ , in the notation of (3.7). Thus  $e$  is determined by specifying elements  $e^{(B,\phi)}$  in  $\mathfrak{X}\{L_{(N_0^\phi, B)}\}_0$  for each  $B, \phi$  as above. But since  $L_{(N_0^\phi, B)}$  is a limit in  $\Theta_{\mathcal{A}}$ , by Remark 3.8, by our assumptions in §1.1 (and §1.8)  $\mathfrak{X}\{L_{(N_0^\phi, B)}\}$  is itself the limit of the corresponding diagram  $\{\mathfrak{X}\{PB_{(\Phi)}\} \xrightarrow{p} \mathfrak{X}\{B_{(\phi)}\}\}$ . Thus  $e^{(B,\phi)}$  is determined by choosing the tautological elements  $\Phi$  in  $\mathfrak{X}\{PB_{(\Phi)}\}_0$  and  $\phi$  in  $\mathfrak{X}\{B_{(\phi)}\}_0$ .

Once we have specified  $\eta : \mathfrak{M}_{\mathcal{A}}T\mathfrak{X}_{\bullet} \rightarrow \mathfrak{X}$ , we set  $\eta_{\mathfrak{X}} : \mathfrak{T}\mathfrak{X}_{\bullet} \rightarrow \mathfrak{X}_{\bullet}$  equal to  $\eta^{\delta}$ . To verify that the square (3.15) commutes, we replace  $\mathfrak{T}\mathfrak{T}\mathfrak{X}_{\bullet} = \mathfrak{M}_{\mathcal{A}}^{\delta}T\mathfrak{T}\mathfrak{X}_{\bullet}$  by  $\mathfrak{M}_{\mathcal{A}}T\mathfrak{T}\mathfrak{X}_{\bullet}$ , and check that the resulting diagram of  $\mathcal{A}$ -mapping algebras commutes by a straightforward diagram chase, again using the fact that  $\mathfrak{M}_{\mathcal{A}}T\mathfrak{T}\mathfrak{X}_{\bullet}$  is free on  $D := T\mathfrak{T}\mathfrak{X}_{\bullet} \in \Theta_{\mathcal{A}}$ .  $\square$

**3.17. Definition.** Given an  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}$ , let  $\mathfrak{V}_{\bullet}^{\delta}$  be the functorial simplicial discrete  $\mathcal{A}$ -mapping algebra-resolution of  $\mathfrak{X}_{\bullet}$  defined by the comonad  $\mathfrak{T}$  (with  $\mathfrak{V}_n^{\delta} = \mathfrak{T}^{n+1}\mathfrak{X}$ ), equipped with the augmentation  $\eta_{\mathfrak{X}} : \mathfrak{V}_0^{\delta} \rightarrow \mathfrak{X}_{\bullet}$ . Set  $\mathfrak{V}_n := \mathfrak{M}_{\mathcal{A}}T\mathfrak{T}^n\mathfrak{X}$  for all  $n \geq 0$ . Because  $\eta_{\mathfrak{X}}$  lifts to a map of  $\mathcal{A}$ -mapping algebras  $\eta : \mathfrak{V}_0 \rightarrow \mathfrak{X}$  (by the proof of Proposition 3.16), the simplicial discrete  $\mathcal{A}$ -mapping algebra  $\mathfrak{V}_{\bullet}^{\delta}$  lifts to a simplicial  $\mathcal{A}$ -mapping algebra  $\mathfrak{V}_{\bullet}$ . Moreover, if we set  $W^0 = T\mathfrak{X}$ , and  $W^n := T\mathfrak{V}_{n-1}$  for  $n \geq 1$ , we obtain a cosimplicial space  $W^{\bullet}$ , with the missing coface map induced by  $\eta$  (so  $d^n : W^{n-1} \rightarrow W^n$  is  $T\mathfrak{T}^{n-1}(\eta)$ ). This  $W^{\bullet}$  is called the *dual Stover resolution* of  $\mathfrak{X}$ , and  $\mathfrak{V}_{\bullet} = \mathfrak{M}_{\mathcal{A}}W^{\bullet}$  by construction.

We now define the *realization functor*  $K : \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{S}_*$  by setting  $K\mathfrak{X} := \text{Tot}(W^{\bullet})$ .

**3.18. Theorem.** *For any simply-connected  $Y \in \mathcal{S}_*$   $K\mathfrak{M}_{\mathcal{A}}Y$  is  $R$ -equivalent to  $Y$ .*

*Proof.* The dual Stover resolution  $W^{\bullet}$  of  $\mathfrak{M}_{\mathcal{A}}Y$  is a weak  $\mathcal{G}$ -resolution of  $Y$  since by Proposition 3.13 it suffices to check that the corresponding simplicial  $\mathcal{A}$ -mapping algebra  $\mathfrak{V}_{\bullet} = \mathfrak{M}_{\mathcal{A}}W^{\bullet}$  is an  $\mathcal{A}$ -mapping algebra-resolution of  $\mathfrak{M}_{\mathcal{A}}Y$ . Since the homotopy spectral sequence for  $W^{\bullet}$  with respect to  $R$  (coinciding with the Bousfield-Kan spectral sequence) converges to  $\pi_* \text{Tot } W^{\bullet}$  (modulo  $R$ ), by [BK2, §6], the result follows (compare [BB2, Proposition 9.23]).  $\square$

In order to apply the general theory above to the homotopy spectral sequence of a cosimplicial space, note that from Proposition 3.13 and [Bou, Theorems 6.2 & 6.5]) we can deduce:

**3.19. Lemma.** *If  $F : \mathcal{S}_* \rightarrow \mathcal{S}_*$  preserves  $\mathcal{A}$ -equivalences, then the homotopy spectral sequences for  $FR^{\bullet}$  and  $FW^{\bullet}$  are isomorphic from the  $E_2$ -term on, where  $X \rightarrow W^{\bullet}$  is as in §3.17, and  $X \rightarrow R^{\bullet}$  is the Bousfield-Kan cosimplicial resolution of [BK1, I, §2],*

In particular, this holds for  $F = \text{Id}$  and  $R = \mathbb{F}_p$  (which yields the usual Adams spectral sequence for  $Y$ ), or for  $F = \text{map}_*(B, -)$  (from which we obtain the spectral sequence for  $\text{map}_*(B, Y)$  – see [BK1, X, §7.5]).

#### 4. CHAIN COMPLEXES AND SPECTRAL SEQUENCES

We now explain how the spectral sequence associated to a chain complex in a model category can be described in terms of higher order chain complexes (cf. [BB3]):

**4.1. Definition.** Let  $\Gamma$  be the category enriched in  $(\text{Set}_*, \wedge)$  with object set  $\{\mathbf{n}\}_{n=-\infty}^\infty$  and a single non-zero arrow  $d_k : \mathbf{k} \rightarrow \mathbf{k} - \mathbf{1}$  for each  $k \in \mathbb{Z}$ , satisfying  $d_{k-1} \circ d_k = o$  for all  $k$ . We denote by  $[\mathbf{m}, \mathbf{n}]$  (for  $-\infty \leq m \leq n \leq \infty$ ) the obvious full subcategories of  $\Gamma$ .

A *chain complex* in a pointed category  $\mathcal{C}$  is a pointed functor  $C_* : \Gamma \rightarrow \mathcal{C}$  (that is, a  $\text{Set}_*$ -enriched functor), an *n-window chain complex* is a pointed functor  $[\mathbf{k} + \mathbf{n}, \mathbf{k}] \rightarrow \mathcal{C}$ . As usual, we write  $\partial_n : C_n \rightarrow C_{n-1}$  for  $C_*(d_n) : C_*(\mathbf{n}) \rightarrow C_*(\mathbf{n} - \mathbf{1})$ , and  $Z_n C_* := \text{Ker}(\partial_n) \subseteq C_n$ , for the *n-cycles* object.

The category of all chain complexes in  $\mathcal{C}$  is denoted by  $\mathcal{C}^\Gamma$ , and similarly we denote by  $\mathcal{C}^{[\mathbf{m}, \mathbf{n}]}$  the category of truncated chain complexes  $(-\infty \leq m \leq n \leq \infty)$ .

**4.2. Remark.** If  $\mathcal{C}$  is a pointed model category, there is an injective model category structure on  $\mathcal{C}^\Gamma$ , (or  $\mathcal{C}^{[\mathbf{m}, \mathbf{n}]}$ ), in which cofibrations and weak equivalences are defined objectwise, and fibrations are determined by the left lifting property (see [Ho, Theorem 5.1.3] or [Hir, §15]). In particular,  $C_* : \Gamma \rightarrow \mathcal{C}$  is *fibrant* if for each  $n \in \mathbb{Z}$ , the map  $C_n \rightarrow \lim_{\text{Hom}_\Gamma(n, i)} C_i$  is a fibration. This means that for every  $n \in \mathbb{Z}$  we have a fibration sequence:

$$(4.3) \quad Z_n C_* \xrightarrow{j_n} C_n \xrightarrow{\partial'_n} Z_{n-1} C_*,$$

where  $\partial_n = j_{n-1} \circ \partial'_n$ , and  $Z_n C_*$  is fibrant.

**4.4. Definition.** A *cochain complex* is a pointed functor  $C^* : \Gamma^{\text{op}} \rightarrow \mathcal{C}$  into some pointed category. However, the concept we shall actually need here is that of an  *$\Omega$ -cochain complex* in a pointed model category  $\mathcal{C}$ : this consist of a sequence of fibrant cochain windows  $(C_{[n]}^* : [\mathbf{n}, \mathbf{0}]^{\text{op}} \rightarrow \mathcal{C})_{n=1}^\infty$ , equipped with a (dimension-wise) weak equivalence of *n-windows*:

$$(4.5) \quad f_{[n]} : \Omega C_{[n]}^* \longrightarrow \tau_n C_{[n+1]}^*$$

for each  $n \geq 1$  (where the truncation functor  $\tau_n$  is induced by the inclusion  $[\mathbf{n}, \mathbf{0}] \hookrightarrow [\mathbf{n} + \mathbf{1}, \mathbf{0}]$ ).

**4.6. Definition.** Let  $B$  be a homotopy cogroup object in a pointed simplicial model category  $\mathcal{C}$ , and  $C_* : [0, \infty] \rightarrow \mathcal{C}$  a fibrant chain complex. For each  $n \geq 0$ , applying the mapping space functor  $\text{map}_{\mathcal{C}}(B, -)$  to the fibration sequence (4.3) yields a fibration sequence

$$(4.7) \quad \text{map}_{\mathcal{C}}(B, Z_n C_*) \xrightarrow{(j_n)_\#} \text{map}_{\mathcal{C}}(B, C_n) \xrightarrow{(\partial'_n)_\#} \text{map}_{\mathcal{C}}(B, Z_{n-1} C_*)$$

in  $\mathcal{S}_*$ , and thus a long exact sequence in homotopy groups:

$$(4.8) \quad \begin{aligned} \dots \longrightarrow [\Sigma^t B, Z_n C_*] &\xrightarrow{(j_n)_*} [\Sigma^t B, C_n] \xrightarrow{(\partial'_n)_*} [\Sigma^t B, Z_{n-1} C_*] \longrightarrow \\ &[\Sigma^{t-1} B, Z_n C_*] \longrightarrow \dots \longrightarrow [B, Z_{n-1} C_*]. \end{aligned}$$

Moreover, for each  $t \geq 0$ ,  $([\Sigma^t B C_n], (\partial_n)_*)_{n=0}^\infty$  is a chain complex of groups (abelian, if  $t \geq 1$ ), and because  $\text{Im}(\partial_{n+1})_* \subseteq \text{Im}(j_n)_* = \text{Ker}(\partial'_n)_\#$ , its homology  $H_n[\Sigma^t B, C_*]$  is a group (abelian, if  $t \geq 1$ ).

A chain complex map  $f : C_* \rightarrow C'_*$  is called a *B-weak equivalence* if for each  $n, t \geq 0$ ,  $f$  induces an isomorphism in  $H_n[\Sigma^t B C_*]$ .

**4.9. Definition.** For  $B$  and  $C_*$  as above, the long exact sequences (4.8), for various  $n \geq 1$ , fit together to yield an exact couple. We call the (homological)

spectral sequence associated to this exact couple the *spectral sequence* of  $C_*$  (with respect to  $B$ ), with

$$E_{m,t}^2 = H_m[\Sigma^t B C_*].$$

We do not address the issue of convergence (or abutment) of this spectral sequence.

The (cohomological) spectral sequence of an  $\Omega$ -cochain complex  $C^*$  (with respect to  $B$ ) is defined analogously, with the indexing taking the shift map (4.5) into account, so that

$$E_2^{m,t} = H^m[\Sigma^t B \Omega^m C_{[m]}^*] \cong H^m \pi_{t-m} \text{map}(B C_{[m]}^*).$$

Here  $H^m[\Sigma^t B \Omega^m C_{[m]}^*]$  is the cohomology of the obvious cochain complex (in groups). A weak equivalence of  $\Omega$ -cochain complexes is defined analogously to §4.6.

**4.10. Definition.** An  $r$ -truncated homological *spectral sequence* is one defined up to and including the  $E^r$ -term, together with the differential  $d^r : E_{t,i}^r \rightarrow E_{t-r-1,t+r}^r$ . We do not require that  $d^r \circ d^r = 0$ , so the  $E^{r+1}$ -term is defined in terms of the  $r$ -truncated spectral sequence only if  $d^r$  is indeed a differential (see [BB1, Definition 4.11]). The cohomological version is similarly defined.

The main example is the  $r$ -truncation of an ordinary spectral sequence (such as that of a (co)chain complex, or a (co)simplicial space – see §5.3 and §5.11).

**4.11. Lemma.** *Let  $B$  be a homotopy cogroup object in a pointed simplicial model category  $\mathcal{C}$ , and  $f : C_* \rightarrow C'_*$  a  $B$ -weak equivalence between fibrant chain complexes in  $\mathcal{C}$ . Then  $f$  induces an isomorphism on spectral sequences from  $E^2$ -term on. Similarly for cochain complexes.*

*Proof.* By Definition 4.6,  $f$  induces an isomorphism on  $E^2$ -terms, and the higher  $E^r$ -terms are the homology of the lower ones.  $\square$

**4.12. Higher order chain complexes.** The notion of a higher order chain complex was introduced in [BB3, §4]. We shall only require the following special case here:

Let  $B$  be a homotopy cogroup object in a pointed model category  $\mathcal{C}$ , and  $C_* : [\mathbf{k} + \mathbf{n}, \mathbf{k} - \mathbf{1}] \rightarrow \mathcal{C}$  a fibrant  $n$ -window chain complex ( $n \geq 1$ ). A *left  $n$ -th order chain complex window* for a homotopy class  $\alpha \in [B, C_{k+n}]_{\mathcal{C}}$  is a sequence  $H_\alpha = (h_i)_{i=0}^n$  of maps  $h_i : B \otimes I^{\otimes i} \rightarrow C_{k+n-i}$ , where  $h_0 : B \rightarrow C_{k+n}$  represents  $\alpha$ , and  $h_i$  is a nullhomotopy for  $\partial_{k+n-i+1} \circ h_{i-1}$  when  $1 \leq i \leq n$ . Here  $B \otimes I^{\otimes i} := (B \otimes I^{\otimes i-1}) \otimes I$ , where  $A \otimes I^0 = A$  and  $A \otimes I$  is a cylinder object for  $A$  in the pointed model category  $\mathcal{C}$  (cf. [Q, I, §2]).

Its *associated value*  $\langle H_\alpha \rangle$  is the homotopy class in  $[\Sigma^n B, C_{k-1}]_{\mathcal{C}}$  of the map induced from  $\partial_k \circ h_n$  by the fact that it maps  $B \otimes \partial(I^{\otimes n})$  to zero.

Given a fibrant  $n$ -window cochain complex  $C^* : [\mathbf{k} + \mathbf{n}, \mathbf{k}]^{\text{op}} \rightarrow \mathcal{C}$  and  $\alpha \in [B, C_k]_{\mathcal{C}}$ , a *left  $n$ -th order cochain complex window*  $H^\alpha$  is defined analogously (with reverse indexing). Its associated value  $\langle H^\alpha \rangle$  lies in  $[\Sigma^n B, C_{k+n+1}]_{\mathcal{C}}$ .

**4.13. Definition.** Given a homotopy cogroup object  $B$  in a pointed model category  $\mathcal{C}$ , a fibrant chain complex  $C_* : [0, \infty] \rightarrow \mathcal{C}$ , and  $n \geq 0$ , an  $n$ -spectral system for  $B$  and  $C_*$  is a collection  $\mathcal{H}_{(B, C_*)}$  of left  $m$ -th order chain complex windows  $H_\alpha$ , indexed by all homotopy classes  $\alpha \in [\Sigma^r B, C_{k+m}]$  for all  $r \geq 0$  and  $k \geq 1$ , such that  $m$  is the maximal value  $\leq n$  for which such an  $m$ -th order chain complex window exists for the given class  $\alpha$ .

An  $n$ -spectral system  $\mathcal{H}^{(B, C^*)}$  for an  $\Omega$ -cochain complex  $C^*$  (with respect to  $B$ ) is defined analogously.

**4.14. Proposition.** *Let  $B$  be a homotopy cogroup object and  $C_*$  a fibrant chain complex in a pointed simplicial model category  $\mathcal{C}$ . Then for each  $n \geq 0$ , the  $(n+1)$ -truncated (homological) spectral sequence (§4.9) is determined by any  $n$ -spectral system  $\mathcal{H}_{(B,C_*)}$  for  $B$  and  $C_*$ .*

*Similarly, the  $(n+1)$ -truncated (cohomological) spectral sequence for an  $\Omega$ -cochain complex  $C^*$  is determined by any  $n$ -spectral system  $\mathcal{H}^{(B,C^*)}$ .*

*Proof.* This follows from the proof of [BB1, Theorem 4.13]: the differential  $d^r$  on a class  $\langle \alpha \rangle \in E_{m,t}^r$  represented by  $\alpha \in [\Sigma^t B C_m]$  is the associated value  $\langle H_\alpha \rangle$  of some left  $r$ -th order chain complex window  $H_\alpha = (h_q)_{q=0}^r$  for  $\alpha$ , because by induction on  $1 \leq q < r$  we can use the indeterminacy in  $E_{m-q,t+q-1}^q = \text{Ker}(d_{m-q,t+q-1}^{q-1}) / \text{Im}(d_{m-2q,t}^{q-1})$  to choose the representative  $h_0 : \Sigma^t B \rightarrow C_m$  for  $\alpha$  so that  $d^q(\langle \alpha \rangle)$  is represented by 0 in  $E_{m-q,t+q-1}^1 = [\Sigma^{t+q-1} B, C_{m-q}]$ . This means that it is null-homotopic, and we choose a null-homotopy  $h_q$  for it.

We need only note that the differentials in the spectral sequence of a tower of fibrations (described by an interlocking system of fibration sequences as in (4.3) or (4.7)) are computed on  $[\alpha]$  as above by choosing an inverse to the connecting homomorphism  $\partial$  of the long exact sequence of the appropriate fibration. Recall that, in general, given a fibration sequence  $F \xrightarrow{i} E \xrightarrow{p} X$  and a map  $f : S^j \rightarrow F$  with  $i_\# [f] = 0$ , a nullhomotopy  $h : S^j \times I \rightarrow E$  for  $i \circ f$  yields a self-nullhomotopy  $p \circ h$  of  $p \circ i \circ f = 0 : S^j \times I \rightarrow X$ , which thus induces a map  $\widetilde{p \circ h} : \Sigma S^j \rightarrow X$  representing  $\partial[f]$ .

In our case,  $F \rightarrow E \rightarrow X$  is

$$\text{map}(\Sigma^N B, Z_{m-q} C_*) \xrightarrow{(j_{m-q})_\#} \text{map}(\Sigma^N B, C_{m-q}) \xrightarrow{(\partial'_{m-q})_\#} \text{map}(\Sigma^N B, Z_{m-q-1} C_*$$

(where  $N = t + q - 1$ ).

By induction, the map  $f : \Sigma^{t+q-1} B \rightarrow Z_{m-q} C_*$  to which we apply the connecting homomorphism is obtained by composing the map  $(\partial'_{m-q-1})_\# [\Sigma^{t+q-1} B, C_{m-q}] \rightarrow [\Sigma^{t+q-1} B, Z_{m-q-1} C_*]$  with the associated value  $\langle (h_i)_{i=0}^q \rangle$  of the chosen left  $q$ -th order chain complex window.  $\square$

In the course of proving the above Proposition we actually showed:

**4.15. Corollary.** *For each  $r \geq 2$  and  $\xi \in E_{m,t}^r$ , the set of homotopy classes  $\beta \in [\Sigma^{t+r-1} B, C_{m-r}]$  representing  $d_{m,t}^r(\xi)$  in  $E_{m-r,t+r-1}^r$  is equal to the set of all associated values  $\langle H_\alpha \rangle$  of an  $n$ -spectral system  $\mathcal{H}_{(B,C_*)}$  for  $B$  and  $C_*$ , for all classes  $\alpha \in [\Sigma^t B, C_m]$  representing  $\xi$ .*

## 5. THE HOMOTOPY SPECTRAL SEQUENCE OF A (CO)SIMPLICIAL SPACE

Next, we explain how the homotopy spectral sequences of both simplicial and cosimplicial spaces fit into the framework described in the previous section.

**5.1. Definition.** Given a simplicial object  $X_\bullet \in s\mathcal{C}$ , over a complete pointed category  $\mathcal{C}$ , recall that its  $n$ -chains object is

$$(5.2) \quad C_n X_\bullet := \{x \in X_n \mid d_i x = * \text{ for } i = 1, \dots, n\}$$

( $n \geq 1$ ), and its  $n$ -cycles object is:

$$Z_n X_\bullet := \{x \in X_n \mid d_i x = * \text{ for } i = 0, \dots, n\},$$

with  $C_0 X_\bullet = Z_0 X_\bullet := X_0$ . The map  $d_0|_{C_n X_\bullet} : C_n X_\bullet \rightarrow Z_{n-1} X_\bullet$  is denoted by  $\mathbf{d}_0^{X_n}$ , and  $(C_n X_\bullet, \mathbf{d}_0^{X_n})_{n=0}^\infty$  is called the *Moore chain complex* for  $X_\bullet$ .

**5.3. The spectral sequence of a bisimplicial set.** For any bisimplicial set  $W_\bullet \in s\mathcal{S}_*$ , Bousfield and Friedlander constructed a first-quadrant spectral sequence of the form

$$(5.4) \quad E_{s,t}^2 = \pi_s \pi_t W_\bullet \Rightarrow \pi_{s+t} \|W_\bullet\| ,$$

where  $\|W_\bullet\| \in \mathcal{S}$  is the realization (that is, the diagonal). The spectral sequence is always defined, but need not always converge (see [BF, Theorem B.5]).

**5.5. Proposition.** *The homotopy spectral sequence for a pointed bisimplicial set  $W_\bullet \in s\mathcal{S}_*$ , for which each  $W_i$  is connected, may be identified with that of a fibrant chain complex in  $\mathcal{T}_*$ , with  $B = S^1$ .*

*Proof.* We may replace  $W_\bullet$  by a weakly equivalent Reedy fibrant simplicial space  $X_\bullet \in s\mathcal{T}_*$  (cf. [Hir, 15.3]). We then have a fibration sequence in  $\mathcal{T}_*$  :

$$(5.6) \quad Z_{n+1}X_\bullet \xrightarrow{j_{n+1}^{X_\bullet}} C_{n+1}X_\bullet \xrightarrow{d_0^{X_{n+1}}} Z_nX_\bullet$$

(see [DKSt2, Prop. 5.7]). Furthermore, by [St, Lemma 2.7], for each  $n \geq 0$  there is a natural isomorphism:

$$(5.7) \quad \pi_* C_n X_\bullet \cong C_n \pi_* X_\bullet ,$$

induced by the inclusion  $C_n X_\bullet \hookrightarrow X_n$ . In [DKSt2, §8.4], Dwyer, Kan, and Stover showed that the Bousfield-Friedlander spectral sequence (5.4) for  $W_\bullet$  is then isomorphic from the  $E^2$ -term on with the spectral sequence of §4.9, associated to the exact couple of (5.6) (that is, to the Moore chain complex  $C_* X_\bullet$ ).  $\square$

**5.8. Lemma.** *The  $(n+2)$ -truncated homotopy spectral sequence of a Reedy fibrant simplicial space  $X_\bullet$  depends only on the simplicial  $n$ -stem  $\mathcal{P}[n]X_\bullet$ .*

*Proof.* Note that  $P^n C_k X_\bullet \rightarrow C_k P^n X_\bullet$  is a weak equivalence, by (5.7), and that  $C_k \text{map}_*(Y, X_\bullet) \cong \text{map}_*(Y, C_k X_\bullet)$  for any  $Y$ , since the Moore chains are a (pointed) limit. Furthermore, the  $(n+1)$ -spectral system  $\mathcal{H}_{(B, C_* X_\bullet)}$  takes values in  $[\Sigma^r B, C_k X_\bullet]$ , and for  $B = S^1$  we have

$$\begin{aligned} P^i \text{map}_*(\Sigma^r B, Y) &\cong \text{map}_*(\Sigma^r B, P^{i+r+1} Y) \\ &\cong \text{map}_*(B, P^{i+r+1} Y) \cong \text{map}_*(B, P^{i+1} Y) \end{aligned}$$

for any  $Y$  and  $i \geq 0$ . Finally, the inclusions  $Z_k X_\bullet \hookrightarrow C_k X_\bullet \hookrightarrow X_n$  induce inclusions  $\text{map}_*(Y, Z_k X_\bullet) \hookrightarrow \text{map}_*(Y, C_k X_\bullet) \hookrightarrow \text{map}_*(Y, X_k)$ , so all choices of simplices in the first two spaces can be made in  $\text{map}_*(Y, X_k)$ . Thus the result follows from Propositions 4.14 and 5.5 (or [BB1, Theorem 4.13]).  $\square$

**5.9. Cosimplicial spaces.** Given a (Reedy) fibrant cosimplicial simplicial set  $W^\bullet \in c\mathcal{S}$ , we think of the (external) cosimplicial direction as *horizontal*, and the (internal) simplicial direction as *vertical*. If  $\Delta^\bullet \in c\mathcal{S}$  denotes the cosimplicial space with  $\Delta^n := \Delta[n]$ , the *total space* of  $W^\bullet$  is  $\text{Tot } W^\bullet := \text{map}_{c\mathcal{S}}(\Delta^\bullet, W^\bullet) \in \mathcal{S}$ . This is the (homotopy) limit of the tower of fibrations (in  $\mathcal{S}$ ):

$$\dots \rightarrow \text{Tot}^{n+1} W^\bullet \xrightarrow{p^{(n+1)}} \text{Tot}^n W^\bullet \xrightarrow{p^{(n)}} \text{Tot}^{n-1} W^\bullet \dots \rightarrow \text{Tot}^0 W^\bullet = W^0 ,$$

where  $\text{Tot}^n W^\bullet := \text{map}_{c\mathcal{S}}(\text{sk}_n \Delta^\bullet, W^\bullet)$  (see [BK1, X, 3.2]).

In fact, for each  $n \geq 1$  there is a fibration sequence:

$$(5.10) \quad \Omega_v^n N^n W^\bullet \xrightarrow{j^n} \text{Tot}^n W^\bullet \xrightarrow{p^{(n)}} \text{Tot}^{n-1} W^\bullet ,$$

having connecting map  $\partial^{n-1} : \Omega_v \text{Tot}^{n-1} W^\bullet \rightarrow \Omega_v^n N^n W^\bullet$  (see [BK1, X, Prop. 6.3]). Here  $\Omega_v$  indicates the vertical re-indexing as for loops in  $\mathcal{S}_*$ :  $(\Omega_v \pi_*^v X)_i = \pi_i^v \Omega X = \pi_{i+1}^v X$ .

**5.11. The homotopy spectral sequence of a cosimplicial space.** Bousfield and Kan defined the *homotopy spectral sequence* of  $W^\bullet$  to be that associated to the tower of fibrations (5.10). It has  $E_1^{n,i} := N^n \pi_i W^\bullet \cong \pi_{i-n} \Omega_v^n N^n W^\bullet$  (see [BK1, X, §6]), with

$$(5.12) \quad E_2^{n,i} = \pi^n \pi_i W^\bullet \implies \pi_{i-n} \text{Tot} W^\bullet \quad (i \geq n \geq 0).$$

**5.13. Proposition.** *The homotopy spectral sequence for a cosimplicial space  $W^\bullet$  for which each  $W^i$  is connected may be identified with that of a  $\Omega$ -cochain complex in  $\mathcal{S}_*$ , again with  $B = S^1$ .*

*Proof.* The normalized chains  $(N^k W^\bullet, \delta_\#^k)_{k=0}^\infty$  form a strict cochain complex in  $\mathcal{S}_*$ . In general,  $(N^k W^\bullet, \delta^k)_{k=0}^\infty$  is not fibrant, but for each  $n \geq 0$ , we can make its looped  $n$ -truncation  $\tau^n \Omega_v^n N^* W^\bullet := (\Omega_v^n N^k W^\bullet, \delta^k)_{k=0}^n$  fibrant, as follows:

Let  $Q^n$  be the pullback of  $\text{Tot}^n W^\bullet \xrightarrow{p^{(n)}} \text{Tot}^{n-1} W^\bullet \xleftarrow{q^n} P \text{Tot}^{n-1} W^\bullet$ , where  $q^n$  is the path fibration, and let  $k^n : Q^n \rightarrow P \text{Tot}^{n-1} W^\bullet$  and  $\ell^n : Q^n \rightarrow \text{Tot}^n W^\bullet$  be the structure maps (also fibrations). Since the fiber of  $\ell^n$  is again  $\Omega_v^n N^n W^\bullet$ , the inclusion  $f^n : \Omega_v^n N^n W^\bullet \hookrightarrow Q^n$  is a weak equivalence, we can replace it by a trivial fibration  $\widehat{f}^n : \widehat{\Omega^n N W^\bullet} \rightarrow Q^n$ . Let  $\widehat{j}^{(n)} := \ell^n \circ \widehat{f}^n : \widehat{\Omega^n N W^\bullet} \rightarrow \text{Tot}^n W^\bullet$  be the composite fibration, with  $\widehat{\partial}^n : \widehat{\Omega \text{Tot}^{n-1}} \hookrightarrow \widehat{\Omega^n N W^\bullet}$  the inclusion of the fiber of  $\widehat{j}^{(n)}$ . Then  $\widehat{f}^n$  induces a trivial fibration on the respective fibers  $\phi^{n-1} : \widehat{\Omega \text{Tot}^{n-1}} \rightarrow \Omega_v \text{Tot}^{n-1} W^\bullet$ , which therefore has a section  $s^{n-1} : \Omega_v \text{Tot}^{n-1} W^\bullet \hookrightarrow \widehat{\Omega \text{Tot}^{n-1}}$  (a trivial cofibration).

All these maps fit into a commuting diagram:

$$\begin{array}{ccccccc}
 & & \Omega_v \text{Tot}^{n-1} W^\bullet & \xrightarrow{\partial^{n-1}} & \Omega_v^n N^n W^\bullet & \xrightarrow{j^{(n)}} & \text{Tot}^n W^\bullet \xrightarrow{p^{(n)}} \text{Tot}^{n-1} W^\bullet \\
 & \nearrow \phi^{n-1} & \downarrow & & \downarrow & & \downarrow \\
 \widehat{\Omega \text{Tot}^{n-1}} & \xrightarrow{s^{n-1}} & \Omega_v \text{Tot}^{n-1} W^\bullet & \xrightarrow{\widehat{\partial}^n} & \widehat{\Omega^n N W^\bullet} & \xrightarrow{\widehat{j}^{(n)}} & \text{Tot}^n W^\bullet \\
 & \searrow & \downarrow & & \downarrow & & \downarrow \\
 & & \widehat{\Omega \text{Tot}^{n-1}} & \xrightarrow{\widehat{\partial}^n} & \widehat{\Omega^n N W^\bullet} & \xrightarrow{\widehat{j}^{(n)}} & \text{Tot}^n W^\bullet
 \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, showing the relationships between the various spaces and maps.)

in which the rows are fibration sequences (with  $\partial^{n-1}$  only defined up to homotopy), and the other maps are weak equivalences.

We therefore define the composite of:

$$(5.14) \quad \Omega_v \widehat{\Omega^n N W^\bullet} \xrightarrow{\Omega_v \widehat{j}^{(n)}} \Omega_v \text{Tot}^n W^\bullet \xrightarrow{s^n} \widehat{\Omega \text{Tot}^n} \xrightarrow{\widehat{\partial}^{n+1}} \widehat{\Omega^{n+1} N W^\bullet}$$

to be  $\widehat{\Omega^n \delta^n}$ , the (looped) differential for  $\widehat{\Omega^* N W^\bullet}$ , and (5.14) is its factorization as a fibration  $\Omega_v \widehat{j}^{(n)}$  followed by a cofibration  $\widehat{\partial}^{n+1} \circ s^n$ . It is readily verified that  $\widehat{\Omega^{n+1} \delta^{n+1}} \circ \Omega_v \widehat{\Omega^n \delta^n}$  vanishes on the nose, since  $\Omega_v (\widehat{j}^{(n)} \circ \widehat{\partial}^n) = 0$ .

Note that because of the successive loopings, we can only make  $(N^k W^\bullet, \delta^k)_{k=0}^\infty$  as a whole into a fibrant cochain complex  $(\Omega_v^\infty \widehat{\Omega^k N W^\bullet}, \Omega_v^\infty \widehat{\Omega^k \delta^k})_{k=0}^\infty$  in the stable case (for an infinite loop space model of the corresponding cosimplicial spectrum).  $\square$



5.15. **Definition.** We call the functorial replacement of

$$\Omega_v \text{Tot}^{n-1} W^\bullet \xrightarrow{\partial^{n-1}} \Omega_v^n N^n W^\bullet \xrightarrow{j^{(n)}} \text{Tot}^n W^\bullet$$

by:

$$\Omega_v \widehat{\Omega^n N W^\bullet} \xrightarrow{\Omega_v \widehat{j}^{(n)}} \Omega_v \text{Tot}^n W^\bullet \hookrightarrow \Omega_v^{n+1} N^{n+1} W^\bullet$$

the *fibrant  $n$ -strictification*, and the composite  $\widehat{\Omega^n \delta^n}$  is called the *fibrant  $n$ -coface map*.

We also have the cosimplicial analogue of Lemma 5.8:

5.16. **Lemma.** *The  $(n+2)$ -truncated homotopy spectral sequence of a Reedy fibrant cosimplicial space  $W^\bullet$  depends only on the cosimplicial  $n$ -stem  $\mathcal{P}[n]W^\bullet$ .*

5.17. *Remark.* This paper is formulated unstably, in terms of the homotopy spectral sequence of a simplicial (or cosimplicial) space. However, we can directly apply the results of Section 4 to the usual construction of the spectral sequence of a tower of (co)fibrations in the stable category (see, e.g., [Ra, §2.1], and compare [BB3]).

## 6. THE DEGREE OF A HOMOTOPY FUNCTOR

Quillen introduced the notion of a total derived functor  $\mathbb{L}F$  in [Q] so as to provide a conceptual framework for the traditional derived functors, which are the homology or homotopy groups of  $\mathbb{L}F$ . In our setting, too, we introduced the total  $n$ -th order left derived functor  $\mathbb{L}^n F$  of Section 2 as a conceptual framework, from which one can extract the data needed to calculate differentials in the relevant spectral sequences.

However, these total higher order derived functors are difficult to use in practice, since constructing the  $\mathcal{A}$ -mapping algebra-resolutions  $\mathfrak{V}_\bullet$  needed to define them requires full knowledge of the topologically enriched theory  $\Theta_{\mathcal{A}}$ , as well as the enriched model  $\mathfrak{M}_{\mathcal{A}}Y$ . Moreover, we see that in the case  $n = 0$  we obtain the ordinary notion of derived functors, which usually depend only on  $\pi_0 \Theta_{\mathcal{A}}$  and  $\pi_0 \mathfrak{M}_{\mathcal{A}}Y$  (which are contained in  $\text{ho}\mathcal{C}$ ).

Thus, to have any chance of calculating the higher versions, two further ingredients are needed:

- (a) We need to extract from  $\mathbb{L}^n F$  the minimal information needed to compute the  $E^{n+2}$ -term of the relevant spectral sequence (see §2.7).
- (b) If possible, we must reduce the data needed to compute  $\mathbb{L}^n F$ ;

For the first requirement, by Propositions 4.14, 5.5, and 5.13 and Theorem 2.8, the information needed is encoded in  $n$ -spectral system of §4.13 – though this does not lend itself readily to an algebraic description.

For the second requirement, we need a further assumption on  $F$ , embodied in the following:

6.1. **Definition.** An  $\mathcal{A}$ -homotopy functor  $F : \mathcal{M}_{\mathcal{A}}^{\text{free}} \rightarrow \mathcal{S}_*$  is said to be of *degree  $k$*  if  $\mathbb{L}^n F$  factors through  $\widehat{\mathbb{L}}^n F : s(n+k)\text{-}\mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{S}_*$  for each  $n \geq 0$  (cf. §1.5). In light of Theorem 2.8, we say that a functor  $G : \mathcal{C} \rightarrow \mathcal{S}_*$  is of degree  $k$  if  $G = F \circ \mathfrak{M}_{\mathcal{A}}$  for  $F$  of degree  $k$ .

If  $F : \mathcal{M}_{\mathcal{A}}^{\text{free}} \rightarrow \mathcal{S}_*$  is an  $\mathcal{A}$ -homotopy functor of degree  $k$ , we define its *reduced  $n$ -th order derived functor* to be  $(\widehat{\mathbb{L}}^n F)Y := \widehat{\mathbb{L}}^n F \mathfrak{V}_\bullet \in \text{Stem}_{\Delta_{\text{op}}}^n$ , where  $\mathfrak{V}_\bullet \rightarrow P^{n+k} \mathfrak{M}_{\mathcal{A}}Y$  is a simplicial resolution in  $P^{n+k} \mathcal{S}_*^{P^{n+k} \Theta_{\mathcal{A}}}$ .

We then have the following version of Proposition 2.6, with essentially the same proof:

**6.2. Proposition.** *If  $F : \mathcal{M}_A \rightarrow \mathcal{S}_*$  is an  $\mathcal{A}$ -homotopy functor of degree  $k$ , the reduced  $n$ -th order left derived functors  $\hat{\mathbb{L}}^n F$  are well defined up to weak equivalence in the resolution model category structure on  ${}_s P^{n+k} \mathcal{M}_A$ , and depend on  $Y \in \mathcal{C}$  only up to  $(n+k)$ - $\mathcal{A}$ -equivalence (that is, weak equivalence of  $P^{n+k} \mathfrak{M}_A$ ).*

Our main theoretical result for this paper may then be summarized as follows;

**6.3. Theorem.** *For  $\mathcal{A} \subseteq \mathcal{C}$  as above,  $G = F \circ \mathfrak{M}_A : \mathcal{C} \rightarrow \mathcal{S}_*$  of degree  $k$ , and  $Y \in \mathcal{C}$  with  $\mathcal{A}$ -resolution  $Y \rightarrow W^\bullet$ , the homotopy spectral sequence for  $GW^\bullet$  is determined up to the  $E_{n+1}$ -term by the reduced  $n$ -th order derived functors of  $F$  applied to  $P^n \mathfrak{M}_A Y$ .*

**6.4. Some examples.** Once more, we defer the case relevant to the Adams spectral sequence to the next section, and first consider some easy examples in the cogroup case for  $\mathcal{C} = \mathcal{S}_*$  and  $\hat{\mathcal{A}} = \{S^n\}_{n=1}^\infty$ :

- 1) the functor  $F = \Omega^k$  is of degree  $k$ , for each  $k > 0$ .
- 2) More generally, the mapping space functor  $F = \text{map}_*(E, -)$  has degree  $\leq d$  if  $E$  is a suspension and  $\dim(E) = d$ .
- 3) The Cartesian product  $\times : \mathcal{S}_* \times \mathcal{S}_* \rightarrow \mathcal{S}_*$  has degree 0.
- 4) The abelianization functor  $\mathbb{Z} : \mathcal{S}_* \rightarrow \mathcal{S}_*$  has degree 0.
- 5) The suspension functor  $\Sigma : \mathcal{S}_* \rightarrow \mathcal{S}_*$  has degree 1, as we can see using Milnor's  $FK$  construction for  $\text{map}_*(S^1, \Sigma K)$  (cf. [Mi]).
- 6) The wedge functor  $\vee : \mathcal{S}_* \times \mathcal{S}_* \rightarrow \mathcal{S}_*$  has degree 0, by Milnor's proof of the Hilton-Milnor Theorem (cf. [Mi]).

## 7. THE ADAMS SPECTRAL SEQUENCE

We now show how Theorem 6.3 applies to spectral sequence for a space  $Y \in \mathcal{S}_*$  with respect to a field  $R$ .

When  $R = \mathbb{F}_p$  (and  $Y$  is simply-connected), this is the mod  $p$  unstable Adams spectral sequence (cf. [Re1, BK2]). Note that if  $Y$  is a finite CW complex, we can calculate a specific differential in the *stable* Adams spectral sequence for the spectrum  $\Sigma^\infty Y$ , by using the *unstable* Adams spectral sequence, applied to an appropriate (finite) suspension of  $Y$  (cf. [BK2, Cor. 5.5]). Since this fits better into our general (co)simplicial framework, we consider only the unstable version here.

We denote the full subcategory of *reduced* simplicial sets (that is, those with a single 0-simplex) by  $\mathcal{S}_*^{\text{red}} \subseteq \mathcal{S}_*$ . Using Definition 6.1, we have:

**7.1. Proposition.** *If  $R = \mathbb{F}_p$  or  $\mathbb{Q}$ ,  $\mathcal{A} = \{K(R, n)\}_{n=1}^\infty$ , and  $B \in \mathcal{S}_*$  is a fixed connected co- $H$ -space, then  $\text{map}(B, -) : \mathcal{S}_*^{\text{red}} \rightarrow \mathcal{S}_*^{\text{red}}$  is an  $\mathcal{A}$ -homotopy functor of degree 0.*

*Proof.* First, let  $W^\bullet$  be a cosimplicial resolution of  $Y$  in the  $\mathcal{A}$ -resolution model category structure on  ${}_c \mathcal{S}_*$ , with each  $W^k$  in  $\mathcal{C}_A$  (either by choosing a specific model for  $W^\bullet$ , or by using a suitable lax version of  $\mathcal{C}_A$  – cf. §1.10). Then the simplicial  $n$ -extended  $\mathcal{A}$ -mapping algebra  $\mathfrak{V}_\bullet := P^n \mathfrak{M}_A W^\bullet$  has  $\mathfrak{V}_k = P^n \mathfrak{M}_A W^k$  in simplicial dimension  $k$ , which is determined by the fixed extended  $n$ - $\mathcal{A}$ -mapping algebra  $P^n \mathfrak{M}_A B$  (and the given simplicial category  $\mathcal{C}_A^\lambda$ , of course – cf. §1.8),

since

$$(7.2) \quad \begin{aligned} \mathfrak{V}_k \left\{ \bigvee_i \Sigma^{t_i} B \right\} &= P^n \left( \prod_i \text{map}(\Sigma^{t_i} B, W^k) \right) \\ &= \prod_i P^n \text{map}(B, \Omega^{t_i} W^k) = P^n(\mathfrak{M}_A B) \left\{ \prod_i \Omega^{t_i} W^k \right\}. \end{aligned}$$

The face maps of  $\mathfrak{V}_\bullet$  are similarly determined by those of  $W^\bullet$  (and the given  $P^n \mathfrak{M}_A B$ ): in fact, they depend only on the simplicial extended  $n$ -dual  $\hat{\mathcal{A}}$ -mapping algebra  $\mathfrak{W}_\bullet := P^n \mathfrak{M}_A W^\bullet$ , since the face  $\mathcal{A}$ -mapping algebra face-map  $(d^j)^* : \mathfrak{V}_{k+1} \rightarrow \mathfrak{V}_k$  is induced by map of simplicial sets  $d^j : W^k \rightarrow W^{k+1}$  via the simplicial composition map

$$(7.3) \quad c : P^n \text{map}(\Sigma^t B, W^k) \times P^n \text{map}(W^k, W^{k+1}) \rightarrow P^n \text{map}(\Sigma^t B, W^{k+1})$$

with  $(d^j)^*(f) = c(f, d_j)$ , and we can apply the (7.2) to rewrite (7.3) as

$$c : (\mathfrak{M}_A B) \{ \Omega^t W^k \} \times \mathfrak{W}^{k+1} \{ \Omega^t W^{k+1} \} \rightarrow (\mathfrak{M}_A B) \{ \Omega^t W^{k+1} \}.$$

Similarly for the degeneracies of  $\mathfrak{V}_\bullet$ .

For an arbitrary resolution  $\mathfrak{V}_\bullet$  of  $P^n \mathfrak{M}_A Y$ , apply Lemma 2.3. The fact that  $\text{map}(B, -)$  is an  $\hat{\mathcal{A}}$ -homotopy functor follows from [BK2, Theorem 12.1] and [BK3, 16.1].  $\square$

Thus we deduce from Theorem 6.3 and Proposition 7.1:

**7.4. Theorem.** *If  $R = \mathbb{F}_p$  or  $\mathbb{Q}$  and  $\hat{\mathcal{A}} = \{K(R, n)\}_{n=1}^\infty$ ,  $B$  is a  $d$ -dimensional finite complex, and  $Y$  is simply-connected then for each  $n \geq 0$ , the  $E_{n+2}$ -term of the  $R$ -Adams spectral sequence for  $\text{map}_*(B, Y)$  is determined by the  $n$ - $\mathcal{A}$ -stem of  $B$  (cf. §1.11) and the  $(n+d)$ - $\mathcal{A}$ -stem of  $Y$ .*

**7.5. Corollary.** *The  $E_{n+2}$ -term of the stable mod  $p$  Adams spectral sequence for the sphere (i.e.,  $Y = B = \mathbf{S}^0$ ) is determined by the  $n$ - $\mathcal{A}$ -stable stem of  $\mathbf{S}^0$ .*

### 7.6. The $E_3$ -term of the Adams spectral sequence.

Our hope is that the approach described in this paper may be applied computationally to higher terms of the Adams spectral sequence, following the method of the first author and Mamuka Jibladze for calculating the  $E_3$ -term of the mod  $p$  stable Adams spectral sequence for a space (or spectrum)  $Y$  (cf. [Ba2, BJ1, BJ2]). Note that this  $E_3$ -term depends on the 1- $\mathcal{A}$ -stem of  $Y$  – that is, the mapping algebra  $P^1 \mathfrak{M}_A Y$  (in the stable range).

**7.7. Definition.** Since the first Postnikov section of an arbitrary topological space  $K$  is modelled algebraically by its fundamental groupoid  $\hat{\pi}_1 K$ , the mapping algebra  $P^1 \mathfrak{M}_A Y$  is equivalent to a *track category* – that is, a category enriched in groupoids (see [Ba1, VI, §3] or [BB2, §2]).

A track category  $\mathcal{E}$  consists of two categories with the same object set  $\mathcal{O}$ : namely,  $\mathcal{E}_0$  (the ordinary category underlying  $\mathcal{E}$ ), and  $\mathcal{E}_1$ , where for every two objects  $x, y \in \mathcal{O}$ ,  $\mathcal{E}_1(x, y)$  is a groupoid, with maps  $f : x \rightarrow y$  in  $\mathcal{E}_0$  as objects, and a set  $\mathcal{E}_1(f, g)$  of (invertible) 2-cells  $H : f \Rightarrow g$  from  $s(H) = f : x \rightarrow y$  to  $t(H) = g : x \rightarrow y$ . The 2-cells induce an equivalence relation on maps in  $\mathcal{E}_0$ , with quotient category  $\text{ho } \mathcal{E}$ , the *homotopy category* of  $\mathcal{E}$ .

7.8. *Remark.* In our case, where  $\mathcal{E} := \hat{\pi}_1 \mathfrak{M}_{\mathcal{A}} Y \simeq P^1 \mathfrak{M}_{\mathcal{A}} Y$ , the homotopy category  $\mathrm{ho} \mathcal{E} \cong \pi_0 \mathfrak{M}_{\mathcal{A}} Y$  is enriched in abelian groups – in fact, it is completely determined by  $H^*(Y; \mathbb{F}_p)$  as an unstable algebra over the mod  $p$  Steenrod algebra  $\mathcal{A}_p$  (see [BB2, §4.2]), or in the stable range, by  $H^*(Y; \mathbb{F}_p)$  as an  $\mathcal{A}_p$ -module.

The track category  $\mathcal{E}$  itself is a linear track extension of  $\mathrm{ho} \mathcal{E}$  by the natural system  $\mathcal{M} := \pi_1 \mathfrak{M}_{\mathcal{A}} Y$ , in the sense of [BW], so it is classified up to weak equivalence by a certain class  $\chi_{\mathcal{E}} \in H_{\mathrm{BW}}^3(\mathrm{ho} \mathcal{E}; \mathcal{M})$  in the third Baues-Wirsching cohomology group of the category  $\mathrm{ho} \mathcal{E}$  (cf. [Ba1, VI, Theorem 3.15]). This may be identified with the 0-th  $k$ -invariant for the simplicially enriched category  $\mathfrak{M}_{\mathcal{A}} Y$ , by [BB2, Theorem 6.5].

Since any cocycle representing  $\chi_{\mathcal{E}}$  yields an equivalent track category, one can use an appropriate choice to construct simpler algebraic models for both track categories  $\tilde{\mathcal{E}} := \hat{\pi}_1 \tilde{\mathcal{M}}^{\mathcal{A}}$  (the secondary Steenrod algebra) and  $\mathcal{E} := \hat{\pi}_1 \mathfrak{M}_{\mathcal{A}} Y$  (the secondary model for  $Y$ ). These are described in [Ba2] as the secondary Hopf algebra  $\mathcal{B}$  (which is a certain *pair algebra*, extending  $\mathcal{A}_p$ ), and the secondary cohomology  $\mathcal{H}^*(Y)$ , which is a  $\mathcal{B}$ -module, extending the usual cohomology  $H^*(Y; \mathbb{F}_p)$ .

7.9. **Secondary derived functors.** In [BJ1], secondary derived functors were defined for an (additive) track category  $\mathcal{E}$  as above: this involves secondary chain complexes  $A_*$ , where each differential  $\partial_i : A_i \rightarrow A_{i-1}$  is in  $\mathcal{E}_0$ , and their successive composites are nullhomotopic by  $\eta_n : \partial_n \circ \partial_{n+1} \Rightarrow 0$  in  $\mathcal{E}_1$ .

In our case, where  $\mathcal{E} = \hat{\pi}_1 \mathfrak{M}_{\mathcal{A}} Y$ , we assume that:

$$A_i \begin{cases} = * & \text{for } i < -1 \\ = Y & \text{for } i = -1 \\ \in \tilde{\mathcal{M}}_{\mathcal{A}} & \text{for } i \geq 0, \end{cases}$$

and that the (ordinary) chain complex  $\pi_0 A_*$  in  $\mathrm{ho} \mathcal{E}$  is an  $\mathcal{A}_p$ -module resolution of the dual of  $H^*(Y; \mathbb{F}_p)$ . This is called a *secondary resolution* of  $Y$ .

Now assume that  $c(Y)^{\bullet} \rightarrow W^{\bullet}$  is a fibrant replacement in the  $\mathcal{A}$ -resolution model category  $c\mathcal{S}_*$  for  $\mathcal{A} = \{K(\mathbb{F}_p, i)\}_{i=1}^{\infty}$  (in the stable range), with each  $W^n$  a finite type  $\mathbb{F}_p$ -GEM, assuming  $Y$  is of finite type, as above. In the  $\mathcal{A}$ -resolution model category  $\mathcal{S}_*^{\Theta_{\mathcal{A}}}$  of §2.2, we obtain a cofibrant simplicial replacement  $\mathfrak{X}_{\bullet} := \mathfrak{M}_{\mathcal{A}} W^{\bullet}$  for  $\mathfrak{M}_{\mathcal{A}} Y$ . In the stable range the simplicial  $\mathcal{A}$ -mapping algebra  $\mathfrak{X}_{\bullet}$  is completely determined by a single simplicial space  $V_{\bullet} := \mathfrak{X}_{\bullet} \{K(\mathbb{F}_p, N)\}$  for  $N \gg 0$ . We can then define a secondary resolution of  $Y$  by setting  $A_n := V_n$  and  $\partial_n := \sum_{i=0}^n (-1)^i d_i$ , as usual.

Note that even though we may assume that each space  $V_n$  is strict abelian group object (in fact, a simplicial  $\mathbb{F}_p$ -vector space), not all maps between such spaces are homomorphisms, so  $\partial_n$  need not be a map of simplicial abelian groups. However, the simplicial identities imply that  $\partial_n \circ \partial_{n+1} \sim 0$ , so we do have a secondary cochain complex window (§4.12). Note that this depends only on the 1-stem of  $W^{\bullet}$  with respect to  $\mathcal{A}$  – that is, on  $P^1 \mathfrak{X}_{\bullet}$ .

In any case,  $\pi_0 W^{\bullet}$  is a free cosimplicial resolution of  $H^*(Y; \mathbb{F}_p)$  (as stable  $\mathcal{A}_p$ -modules) – so it can be used to calculate the  $E_3$ -term of the Adams spectral sequence for  $Y$ . Thus by [BJ1, Theorem 7.3], the secondary differential defined in [BJ1, Theorem 6.9] coincides with the differential  $d_2$  described in [BB1, Proposition 3.12].

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